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ELEMENTS

OF

DESCRIPTIVE GEOMETRY

PART I

ORTHOGRAPHIC PROJECTIONS

BY

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PREFACE

CHURCH'S "Elements of Descriptive Geometry" was originally published in 1864. The preface to the first edition states: "Without any effort to enlarge or originate, the author has striven to give, with a natural arrangement and in clear and concise language, the elementary principles and propositions of this branch of science, of so much interest to the mathematical student, and so necessary to both the civil and military engineer."

Professor Church succeeded so well in his efforts to produce a practical and well-adapted treatise that it has continued in use as a text-book for more than forty years in the United States Military Academy and in many other academies, technical schools, and colleges. This long-continued use of the book speaks well for its high intrinsic excellence.

During the last few years, however, there have taken place many changes in the methods of teaching the subject, and in the problems required. To meet these new demands the present volume is issued. In its preparation much of Professor Church's text has been used, and his concise and lucid style has been preserved.

Among the salient features of the present work are the following:

The *figures* and *text* are included in the same volume.

General cases are preferred to special ones.

A sufficient number of problems are solved in the *third angle* to familiarize the student with its use.

A treatment of the *profile plane* of projection is introduced.

Many *exercises for practice* have been introduced.

Several *new problems* have been added.

The *old figures have been redrawn*, and many of them have been improved.

Several of the more difficult elementary problems have been illustrated by *pictorial views*.

In the treatment of *curved surfaces*, all problems relating to single-curved surfaces are taken up first, then those relating to warped surfaces, and finally those relating to surfaces of revolution. Experience proves this order to be a logical one, as we here proceed "from the simple to the more complex." Also the student is more quickly prepared for drawing-room work on intersections and developments; and in case it is desired to abbreviate the course by omitting warped surfaces, the remaining problems will be found to be consecutively arranged.

The writer here wishes to acknowledge his indebtedness to the many teachers who have aided him with valuable advice and suggestions in relation to this work. In particular his thanks are due to his esteemed colleagues, Professor H. J. Goulding and Mr. D. E. Foster of the University of Michigan, for their careful reading and correction of the manuscript.

G. M. B.

MAY 14, 1910.

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PART I

ORTHOGRAPHIC PROJECTIONS

Preliminary Definitions

1. Geometry enables us to determine unknown magnitudes, relationships, and forms from those which are known. There are in general two methods of solution for any given problem; namely, the *analytical* and the *graphical*. In the former we arrive at our results by calculation; in the latter we make drawings which represent graphically the true relationships between the points, lines, and surfaces under consideration, and arrive at our results without calculation.

2. Graphics. If the problem relates to points and lines lying in only one plane, the graphical solution may be reached by a simple application of the principles of Geometrical Drawing, or *Plane Graphics.*

If the problem relates to magnitudes not in the same plane, the graphical solution would require an application of the principles of Descriptive Geometry, or the *Graphics of Space.*

3. Descriptive Geometry is that branch of Mathematics which has for its object the explanation of the methods of representing by drawings:

First. All geometrical magnitudes.

Second. The solution of problems relating to these magnitudes in space.

These drawings are so made as to present to the eye, situated at a particular point, the same appearance as the magnitude or object itself, were it placed in the proper position.

The representations thus made are the *projections* of the magnitude or object.

The planes upon which these projections are usually made are the *planes of projection.*

The point at which the eye is situated is the *point of sight.*

4. Projections. When the point of sight is in a perpendicular drawn to the plane of projection through any point of the drawing, and at an infinite distance from this plane, the projection is *orthographic.*

When the point of sight is within a finite distance of the drawing, the projection is *scenographic*, and is commonly called the *perspective* of the magnitude or object.

If a straight line be drawn through a given point and the point of sight, the point in which this line pierces the plane of projection will present to the eye the same appearance as the point itself, and will therefore be the projection of the point on this plane. The line thus drawn is the *projecting line of the point.*

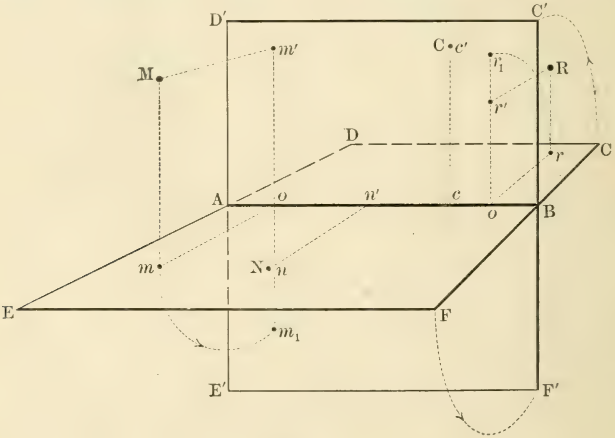

Fig. 1.

5. In the **orthographic projection**, since the point of sight is at an infinite distance, the projecting lines drawn from any points of an object of finite magnitude to this point, *will be parallel to each other and perpendicular to the plane of projection.*

In this projection two planes are used, at right angles to each other, the one *horizontal* and the other *vertical*, called respectively the *horizontal* and the *vertical plane of projection.*

In Fig. 1, let the planes represented by CDEF and C′D′E′F′ be the two planes of projection, the first the *horizontal* and the second the *vertical.*

Their line of intersection AB is the *ground line.*

These planes form by their intersection four dihedral angles. The *first angle* is above the horizontal and in front of the vertical plane. The *second* is above the horizontal and behind the vertical. The *third* is below the horizontal and behind the vertical. The *fourth* is below the horizontal and in front of the vertical.

REPRESENTATION OF POINTS, LINES, AND PLANES

6. Representation of points. Let M, Fig. 1, be any point in space. Through it draw Mm perpendicular to the horizontal, and Mm' perpendicular to the vertical plane; m will be the projection of M on the horizontal, and m' that on the vertical plane (Art. 5). Hence, the *horizontal projection of a point* is the foot of a perpendicular through the point to the horizontal plane; and the *vertical projection of a point* is the foot of a perpendicular through it to the vertical plane.

The lines Mm and Mm' are the horizontal and vertical *projecting lines* of the point.

7. PROPOSITION I.[*] **The distance of a point from the horizontal plane is equal to the distance of its vertical projection from**

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[*] It is important that the student should be able to state and prove all propositions given in this book.

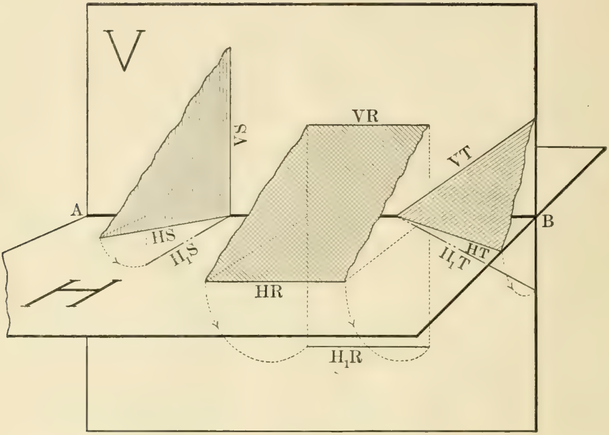


Fig. 2.

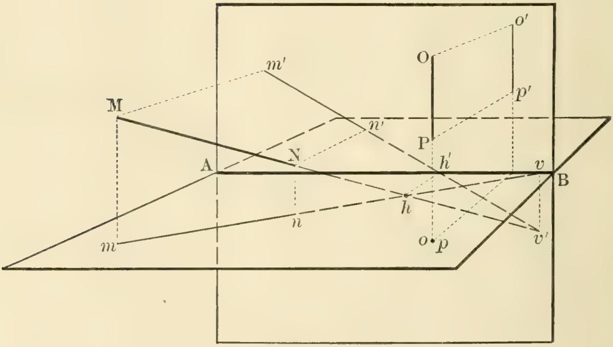


Fig. 3.

the ground line ; and the distance of the point from the vertical plane is equal to that of its horizontal projection from the ground line.

Through the lines Mm and Mm', Fig. **1**, pass a plane. It will be perpendicular to both planes of projection, since it contains a straight line perpendicular to each, and therefore perpendicular to the ground line AB. It intersects these planes in the lines mo and $m'o$, both perpendicular to the ground line at the same point, forming the rectangle Mo. By an inspection of the figure it is seen that $Mm = m'o$, and $Mm' = mo$.

8. Proposition II. **If the two projections of a point are given, the point is completely determined.** For if at the horizontal projection m, Fig. 1, a perpendicular be erected to the horizontal plane, it will contain the point M. A perpendicular to the vertical plane at m' will also contain M ; hence the point M is determined by the intersection of these two perpendiculars.

If N be in the horizontal plane, $Nn = 0$, and the point is its own horizontal projection. The vertical projection will be in the ground line at n'. Similarly, if C be in the vertical plane, it will be its own vertical projection, and its horizontal projection will be in the ground line at c.

If the point be in the ground line, it will be its own horizontal and also its own vertical projection.

9. Representation of planes. Let HT–VT, Fig. 2, be a plane, oblique to the ground line, intersecting the planes of projection in the lines HT and VT respectively. Its intersection with the horizontal plane is the *horizontal trace* of the plane, and its intersection with the vertical plane is the *vertical trace.*

10. Proposition III. **If the two traces of a plane are given, the plane is completely determined.** Why?

11. Representation of straight lines. Let MN, Fig. 3, be any straight line in space. Through it pass a plane Mmn perpendicular to the horizontal plane; *mn* will be the horizontal, and *vv′*, perpendicular to the ground line, the vertical trace of this plane. Also through MN pass a plane M*m′n′* perpendicular to the vertical plane; *m′n′* will be its vertical, and *h′h* its horizontal trace. The traces *mn* and *m′n′* are the projections of the line, the points *m*, *m′*, *n*, *n′* being the projections of the extremities of the line. Hence, the *horizontal projection* of a straight line coincides with the *horizontal trace of a plane passed through the line perpendicular to the horizontal plane;* and the *vertical projection* of a straight line coincides with the *vertical trace of a plane through the line perpendicular to the vertical plane.*

The planes M*mn* and M*m′n′* are respectively the *horizontal* and the *vertical projecting planes* of the line.

12. PROPOSITION IV. **The two projections of a straight line being given, the line will in general be completely determined;** for if through the horizontal projection we pass a plane perpendicular to the horizontal plane, it will contain the line; and if through the vertical projection we pass a plane perpendicular to the vertical plane, it will also contain the line. The intersection of these planes must, therefore, be the line. Hence we say *a straight line is given by its projections.*

13. The projections *mn* and *m′n′* are also manifestly made up of the projections of all the points of the line MN. Hence:

PROPOSITION V. **If a straight line pass through a point in space, its projections will pass through the projections of the point.** Likewise,

PROPOSITION VI. **If, in either plane, the projections of any two points of a straight line are given, the straight line joining these projections will be the projection of the given line.**

14. Propositions relating to the Point, Line, and Plane.

PROPOSITION VII. **The two traces of a plane must intersect the ground line at the same point.** For if they should intersect it at different points, the plane would intersect it in two points, which is impossible.

PROPOSITION VIII. **If a plane is parallel to the ground line, its traces must be parallel to the ground line** (Fig. 2). For if they are not parallel, they must intersect it; in which case the plane would have at least one point in common with the ground line, which is contrary to the hypothesis.

PROPOSITION IX. **If a plane is parallel to either plane of projection, it will have but one trace, which will be on the other plane, and parallel to the ground line.**

PROPOSITION X. **If a plane is perpendicular to the horizontal plane, its vertical trace will be perpendicular to the ground line,** as VS in Fig. 2. For the vertical plane is also perpendicular to the horizontal plane; hence the intersection of the two planes, which is the *vertical trace*, must be perpendicular to the horizontal plane, and therefore to the ground line which intersects it.

Likewise if a plane is perpendicular to the vertical plane, its horizontal trace will be perpendicular to the ground line.

PROPOSITION XI. **If a plane simply pass through the ground line, its position is not determined.**

PROPOSITION XII. **If two planes are parallel, their traces on the same plane of projection are parallel, for these traces are the intersections of the parallel planes by a third plane.**

But if two planes are *perpendicular* to each other, their traces on the same plane will *not* in general be perpendicular. Will they ever be? If so, when?

PROPOSITION XIII. **If a straight line is perpendicular to either plane of projection, its projection on that plane will be a point, and its projection on the other plane will be perpendicular to the ground line.** See OP, Fig. 3.

PROPOSITION XIV. **If a line is parallel to either plane of projection, its projection on that plane will evidently be parallel**

and equal to the line itself, and its projection on the other plane will be parallel to the ground line. See Fig. 4.

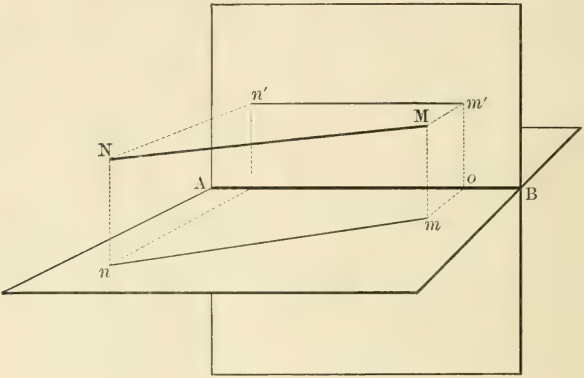


Fig. 4.

Also, if a straight line is parallel to both planes of projection or to the ground line, both projections will be parallel to the ground line.

Proposition XV. **If a line lies in either plane of projection, its projection on that plane will be the line itself, and its projection on the other plane will be in the ground line.** Thus in Fig. 5, RS in the vertical plane is its own vertical projection, and *rs* in the ground line is its horizontal projection.

Proposition XVI. **If the two projections of an unlimited straight line are perpendicular to the ground line, the line is undetermined,** as the two projecting planes coincide, forming only one plane, and do not by their intersection determine the line as in Art. 12.

All unlimited lines in this plane will have the same projections. Thus, in Fig. 5 *mn* and *m′n′* are both perpendicular to

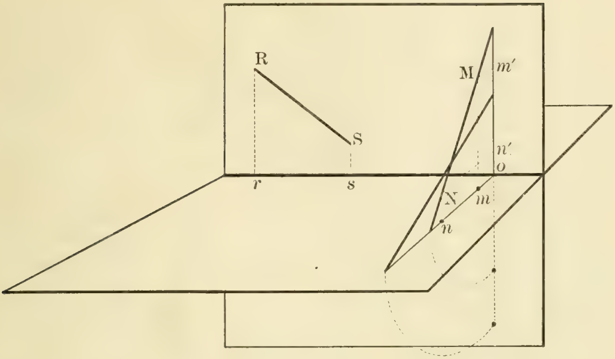


FIG. 5.

the ground line; and any line in the plane MN*o* will have these for its projections.

PROPOSITION XVII. **If, however, the projections of two points of the line are given, the line will then be determined;** that is, if *mm′* and *nn′* are given, the two points, M and N, will be determined, and, of course, the straight line which joins them.

PROPOSITION XVIII. **All lines and points situated in a plane perpendicular to either plane of projection, will be projected on this plane in the corresponding trace of the plane.**

PROPOSITION XIX. **If two straight lines are parallel, their projections on the same plane will be parallel.** For their projecting planes are parallel, since they contain parallel lines and are perpendicular to the same plane; hence their traces will be parallel (Prop. XII), and these traces are the projections.

If two lines are *perpendicular* to each other, their projections on the same plane will *not* in general be perpendicular; but,

PROPOSITION XX. **If two straight lines are perpendicular to each other, and one of them is parallel to one of the planes**

of projection, their projections on that plane will be perpendicular. For the projecting plane of the line which is not parallel to the plane of projection is perpendicular to the second line, and also to its projection, since this projection is parallel to the line itself (Prop. XIV); and since this projection is perpendicular to this projecting plane, it is perpendicular to the trace of this projecting plane, which is the projection of the first line.

PROPOSITION XXI. **Every straight line of a plane, not parallel to the horizontal plane of projection, will pierce it in the horizontal trace of the plane, and if not parallel to the vertical plane, will pierce it in the vertical trace.** Why?

PROPOSITION XXII. **If a straight line lies in a plane and is parallel to the horizontal plane, its horizontal projection will be parallel to the horizontal trace of the plane, and its vertical projection will be parallel to the ground line.** Prove this.

What will be true of the projections of a straight line that is parallel to the vertical plane?

PROPOSITION XXIII. **If two intersecting lines are each parallel to the same plane of projection, the angle between their projections on that plane will be equal to the angle between the lines.** Prove this.

PROPOSITION XXIV. **If a straight line is perpendicular to a plane, its projections will be respectively perpendicular to the traces of the plane.** For if the line PQ, Fig. 6, is perpendicular to the plane MR, the horizontal projecting plane of the line is perpendicular to the given plane MR, since it contains a line, PQ, perpendicular to it. This projecting plane is also perpendicular to the horizontal plane (Art. 11). It is therefore perpendicular to the intersection SR, of these two planes, which is the horizontal trace of the given plane. Hence *pq*, the *horizontal projection of the line*, which is a line of this projecting plane, must be perpendicular *to the horizontal trace.*

In the same way it may be proved that the *vertical projection* of the line will be perpendicular *to the vertical trace.*

PROPOSITION XXV. *Conversely,* **If the projections of a straight line are respectively perpendicular to the traces of a plane, the line will be perpendicular to the plane.** For if through the horizontal projection of the line its horizontal projecting plane be passed, it will be perpendicular to the horizontal trace of the given plane, and therefore perpendicular to the plane. In the same way it may be proved that the vertical projecting plane of the line is perpendicular to the given plane; therefore the intersection of these two planes, which is the *given line*, is perpendicular to the given plane.

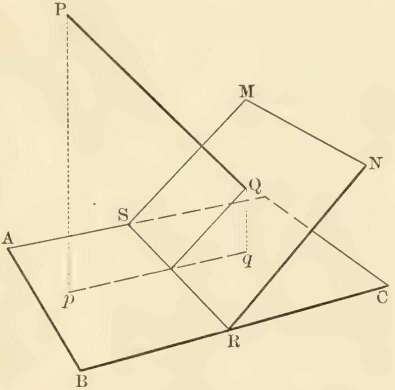


FIG. 6.

Is the above proposition true when the plane is parallel to the ground line?

But if a straight line is *parallel* to a plane, the projections of the line will *not* in general be parallel to the traces of the plane. Will they ever be? If so, under what conditions?

15. Rotation of the horizontal plane. In order to represent both projections of an object on the same sheet of paper or plane, after the projections are made as in the preceding articles, the horizontal plane is rotated about the ground line as an axis until it coincides with the vertical plane, that portion of the horizontal plane which is in front of the ground line falling below it in the position ABF′E′, Fig. 1, and that part which is back of the ground line coming up in the position ABC′D′.

In this new position of the planes it will be observed that, the planes being regarded as indefinite in extent, all that part of the plane of the paper which is below the ground line will represent not only that part of the vertical plane which is below the ground line, but also that part of the horizontal plane which is in front of the ground line ; while the part above the ground line represents that part of the vertical plane which is above the ground line, and also that part of the horizontal plane which is back of the ground line.

16. After the horizontal plane is rotated as in the preceding article, the point m, in Fig. 1, will take the position m_1 in the line $m'o$ produced, and the two projections m_1 and m' will then be in the same straight line, perpendicular to the ground line. Hence :

Proposition XXVI. **The two projections of the same point must be in the same straight line, perpendicular to the ground line.**

Proposition XXVII. **If a point is in the first angle, its horizontal projection will be below, and its vertical projection above, the ground line.**

A point in the second angle will have both projections above the ground line, as R, Fig. 1.

A point in the third angle will have its horizontal projection above, and its vertical projection below, the ground line. Why?

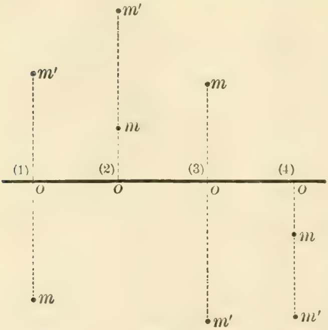


Fig. 7.

A point in the fourth angle will have its projections, where?

Conversely, we can tell in what angle a given point lies by noticing the location of its projections with respect to the ground line.

Proposition XXVIII. **If any two lines intersect, the straight line joining the points in which their projections intersect must be perpendicular to the ground line.** Prove this.

17. Notation to be used in the description of drawings. A point in space will be designated by some capital letter, as M. Its horizontal projection will be designated by the corresponding lower-case letter, as m; and its vertical projection by the same letter *primed*, as m'. We may speak of the point itself either as the point (mm') or simply as the point M.

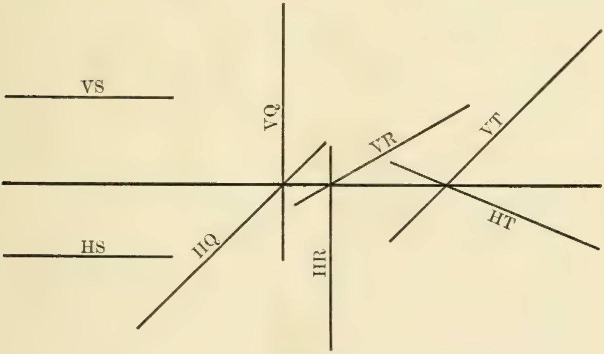


Fig. 8.

A plane will be designated by some capital letter, as S, T, R. Its horizontal trace will be designated, as in Figs. 2 and 8, by a capital H placed before the letter of the plane, as HS, HT, etc. Its vertical trace will be designated by a capital V placed before the letter of the plane, as VS, VT, etc.

Lines given by their projections, as in Fig. 9, will be described as the line (mn, $m'n'$), the letters on the horizontal projection being first in order, or simply the line MN.

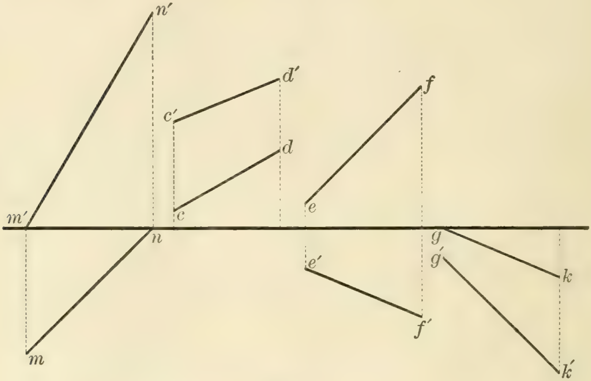


FIG. 9.

The planes of projection will often be described by the capitals H and V; H denoting the horizontal, and V the vertical, plane.

18. The projections of the same point will be connected by a dotted line, thus ....

Traces of planes which are given or required, when they can be seen from the point of sight, — that is, when the view is not obstructed by some intervening opaque object, — are drawn full. When not seen, or when they are the traces of auxiliary planes, not the projecting planes of straight lines, they will be drawn dashed and dotted, thus:

——— ·· ——— ·· ——— ·· ——— ·· ——— ·· ———

Lines, or portions of lines, either given or required, when seen, will have their projections full. When not seen, or auxiliary, their projections will be dashed, thus :

— — — — — — — — — —

In the construction of problems, the planes of projection and all auxiliary surfaces will be regarded as transparent.

All lines or surfaces are regarded as indefinite in extent, unless limited by their form, or a definite portion is considered for a special purpose. Thus the ground line and projections of lines in Fig. 9 are supposed to be produced indefinitely, the lines delineated simply indicating the directions.

EXERCISES FOR PRACTICE

Let the following examples be treated in accordance with Propositions I to XXVIII.*

Ex. 1. State in what angle each of the points shown in Fig. 10 is located, and whether the point is nearer H or V.

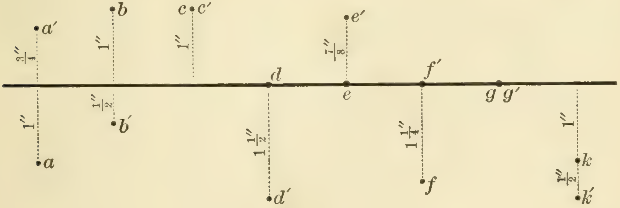


FIG. 10.

Ex. 2. Show the projections of the following points properly lettered and with distances given.

The pt. A, 1″ behind V, $1\frac{1}{2}$″ below H.

The pt. B, lying in V, 1″ below H.

The pt. C, 3″ in front of V, 1″ above H.

The pt. D, 1″ behind V, lying in H.

---

* In these examples, ‖ is to be read *parallel ;* ⊥, *perpendicular;* ∠, *angle,* and ″, *inches.* A *line* will be understood to be straight, unless otherwise stated.

The pt. E, 2″ behind V, $1\frac{1}{2}''$ below H.
The pt. F, 1″ in front of V, 1″ below H.
The pt. G, lying in V, 2″ above H.
The pt. J, 1″ in front of V, lying in H.
The pt. K, lying in V, lying in H.
The pt. L, in 3d angle, 1″ from H, 2″ from V.
The pt. M, in 2d angle, 3″ from H, 2″ from V.
The pt. N, in 1st angle, $1\frac{1}{2}''$ from H, $3\frac{1}{2}''$ from V.
The pt. P, in 4th angle, 4″ from H, 1″ from V.

Ex. 3. If the H projection of a line is || to the ground line, what conclusion do you draw (*a*) as to the *position* of the line? (*b*) as to its intersection with the V plane? (*c*) as to the V trace of any plane passed through the line?

Ex. 4. If a straight line lies in a given plane, what conclusion do you draw as to the points where the line pierces the H and V planes respectively?

Ex. 5. If the H trace of a plane is ⊥ to the ground line, what conclusion do you draw (*a*) as to the *position* of the plane? (*b*) as to the ∠ between the plane and the H plane? (*c*) as to the V projection of any line lying in the plane? (*d*) as to the ∠ between the plane and the V plane?

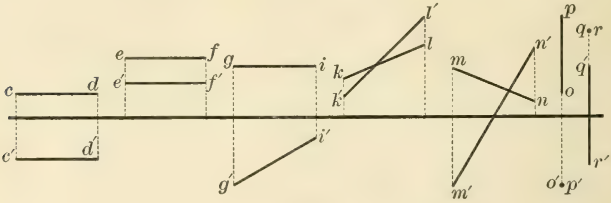


FIG. 11.

Ex. 6. Describe the situation of the lines in Fig. 11 with respect to the *ground line*, the *planes of projection*, and the *angles*.

Ex. 7. Show the projections of the lines

AB, ‖ to H, ‖ to V, in 3d angle.

CD, ‖ to H, ⊥ to V, in 2d angle.

EF, ‖ to H, inclined to V, in 1st angle.

GH, inclined to H, ‖ to V, in 1st angle.

JK, ⊥ to H, ‖ to V, in 2d angle.

MN, inclined to H, inclined to V, in 2d angle.

OP, inclined to both planes of projection and in a plane perpendicular to the ground line, in 1st angle.

QR, inclined to V, and lying in H beyond the ground line.

ST, inclined to H, and lying in V above the ground line.

UV, lying in both H and V.

Ex. 8. Construct the projections of two lines, AB and AC, intersecting in A, one ‖ to H, the other ‖ to V.

Ex. 9. Show the projections of a line joining a point A in the 2d angle with a point B in the 3d angle.

Ex. 10. Show the projections of a line joining a point C in the 4th angle with a point D in the 1st angle.

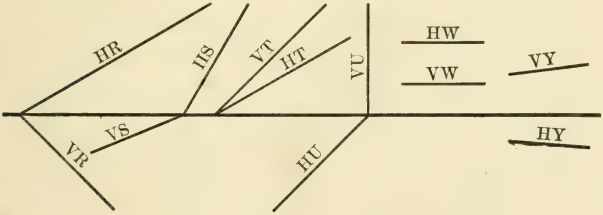


Fig. 12.

Ex. 11. Describe the situation of the planes in Fig. 12 with respect to the *ground line*, the *planes of projection*, and the *angles*.

Ex. 12. Represent by their traces the planes:

A, ⊥ to both H and V.

B, inclined to H; ⊥ to V.

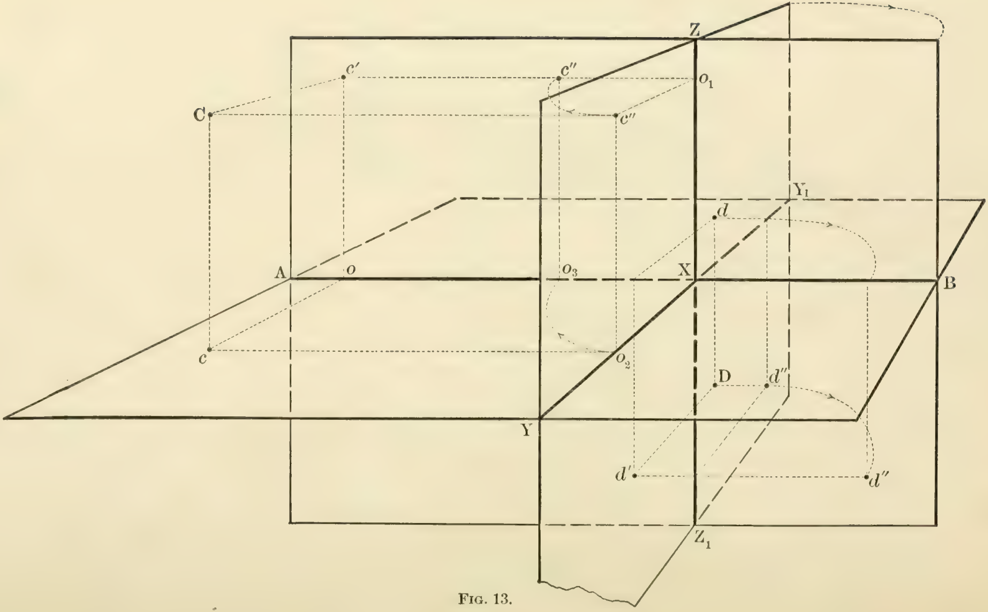


FIG. 13.

C, ⊥ to H; inclined to V.
D, inclined to both H and V.
E, ⊥ to H; ∥ to V.
F, ∥ to H; ⊥ to V.
G, ∥ to the ground line, but not passing through it.
K, containing the ground line.

The Profile Plane of Projection

19. While most problems in descriptive geometry can be solved by means of two planes of projection, it is sometimes convenient to make use of a third plane perpendicular to both H and V, and called the *profile*, or P, *plane of projection.* It is usually placed to the right of the object; and in order to represent on a single sheet of paper all three projections of the object, the profile plane is rotated about its vertical trace as an axis until it coincides with the V plane.

The direction of rotation is usually such as to bring the P and V projections of the object on opposite sides of the profile trace. This practice is almost universal for objects in the *second* or *third angle*, but when objects are placed in the first angle, practice differs. For the purpose of uniformity in the present text, as well as for general convenience, the direction of rotation will be such that the portion lying in front of the V plane falls to the *left* of its V trace, while the portion lying behind the V plane falls to the *right* of the same.

It will be noticed that the ground line and the profile axis cross at right angles and divide the plane of the paper into four *quadrants.*

20. Proposition XXIX. **If a point is in the first angle, its P projection will appear in the upper left-hand quadrant. If a point is in the second angle, its P projection will appear in the upper right-hand quadrant. If in the third angle, its P projection**

appears in the lower right-hand quadrant; and if in the fourth angle, its P projection is in the lower left-hand quadrant.

An examination of Fig. 13 will show the above to be true, and also the following:

21. PROPOSITION XXX. **The distance of the P projection of a point from the ground line is equal to the distance of its V projection from the ground line, and its distance from the profile axis ZZ_1 is equal to the distance of the H projection from the ground line.**

Hence, to determine the profile projection of a point C when its H and V projections are known, we draw through the V projection an indefinite line $c'o_1$ (Figs. 13 and 14) parallel to the ground line. The P projection will lie somewhere in this line. With X as a center and the distance *co* as a radius, draw a quarter circle o_2o_3 in a *clockwise* direction. At the point o_3

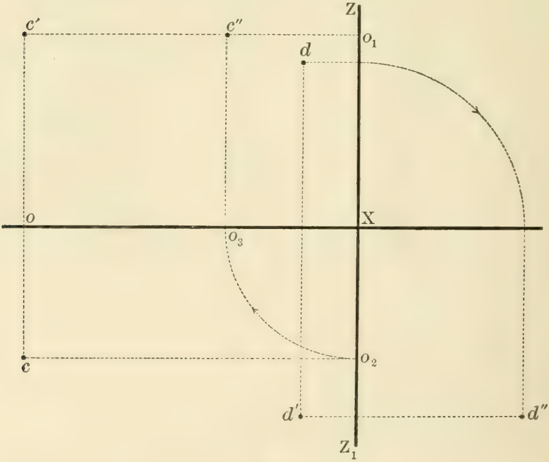


FIG. 14.

erect a perpendicular to the ground line intersecting $c'o_1$ in c'', the required profile projection of C. Similarly for the point D in the third angle.

Ex. 13. Let the student assume points and lines in each of the four angles, and then find their profile projections.

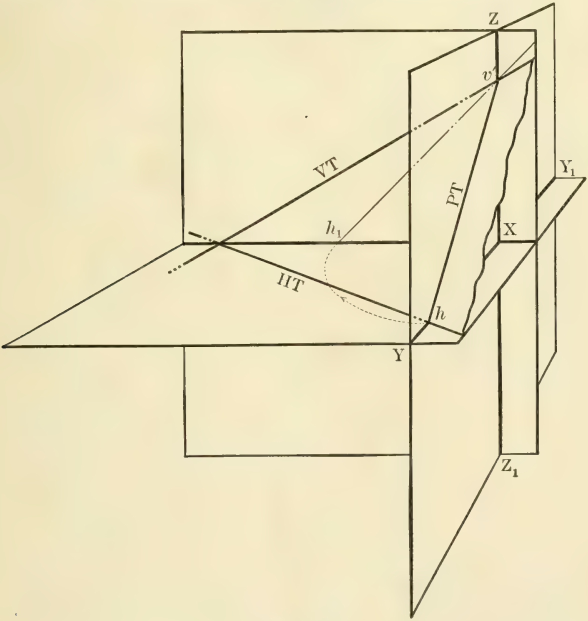


FIG. 15.

22. The profile trace of a plane. In Fig. 15, let T be any oblique plane. Its intersection hv' with the P plane of projection is its profile trace. When the profile plane has been brought into coincidence with V (Art. 19), the point h, where the P and H traces meet, will revolve to the position h_1 in the ground line. The point v', where the P and V traces meet, will not change. Hence the profile trace will take the position h_1v'.

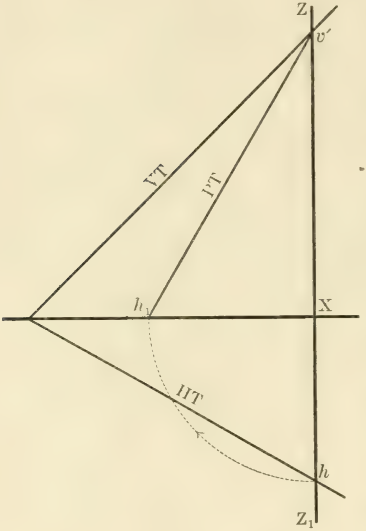


FIG. 16.

To determine the profile trace of a plane, having given the H and V traces, we proceed as follows: With X as center, Fig. 16, and Xh as radius, strike the quarter circle hh_1 in a *clockwise* direction and join h_1 with the point v' where the V trace cuts the profile axis.

Ex. 14. Assume planes such as R, S, T, U, and W, Fig. 12, and find the profile trace of each.

Ex. 15. Assume lines such as GI, KL, QR, and MN, Fig. 11, and find their profile projections.

CONSTRUCTION OF ELEMENTARY PROBLEMS RELATING TO THE POINT, LINE, AND PLANE

23. Having explained the manner of representing with accuracy points, planes, and straight lines, we are now prepared to represent the solution of a number of important problems relating to these magnitudes in space.

In every problem certain points and magnitudes are given, from which certain other points or magnitudes are to be constructed.

Let a straight line be first drawn on the paper to represent the ground line; then assume, as in Arts. 16, 17, and 18, the representations of the given objects. The proper solution of the problem will now consist of two distinct parts. The first is a clear statement of the principles and reasoning to be employed in the construction of the drawing. This is the *analysis* of the problem. The second is the construction, in proper order, of the different lines which are used and required in the problem. This is the *construction* of the problem.

24. Problem 1. **To find the points in which a given straight line pierces the planes of projection.**

Let AB, Fig. 17, be the ground line, and $(mn, m'n')$, or simply MN, the given line.

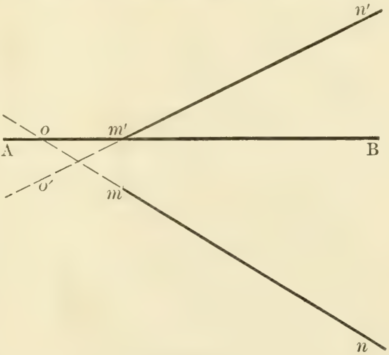


Fig. 17.

First. To find the point in which this line pierces the horizontal plane.

Analysis. Since the required point is in the horizontal plane, its vertical projection is in the ground line (Art. 8); and since the point is in the given line, its vertical projection will be in the vertical projection of this line (Prop. V, Art. 13); hence it must be at the intersection of this vertical projection with the ground line. The horizontal projection of the required point must be in a straight line drawn through its vertical projection, perpendicular to the ground line (Prop. XXVI, Art. 16), and also in the horizontal projection of the given line; hence it will be at the intersection of these two lines. But the point being in the horizontal plane is the same as its horizontal projection (Art. 8); hence the rule:

Produce the vertical projection of the line until it intersects the ground line; at the point of intersection erect a perpendicular to the ground line and produce it until it intersects the horizontal projection of the line; this point of intersection is the required point.

Construction. Produce $m'n'$ to m'; at m' erect the perpendicular $m'm$, and produce it to m. This is the required point.

Second. In the above analysis, by changing the word "vertical" into "horizontal," and the reverse, we have the analysis and rule for finding the point in which the given line pierces the vertical plane.

Construction. Produce mn to o; at o erect the perpendicular oo', and produce it to o'. This is the required point.

Ex. 16. Assume lines similarly situated to KL and MN, Fig. 11, and find where each pierces the H and V planes.

25. PROBLEM 2. **To find the length of a straight line joining two given points in space.**

Let AB, Fig. 18, be the ground line, and (mm') and (nn') the two given points.

Analysis. Since the required line contains the two points, its projection must contain the projections of the points (Prop. VI, Art. 13). Hence, if we join the horizontal projections of the points by a straight line, it will be the *horizontal projection* of the line; and if we join the vertical projections of the points, we shall have its *vertical projection.*

If we now revolve the horizontal projecting plane of the line about its horizontal trace until it coincides with the horizontal plane, and find the revolved position of the points, and join them by a straight line, it will be the required distance, since the points do not change their relative position during the revolution.

Construction. Draw mn and $m'n'$. MN will be the required line.

Now revolve its horizontal projecting plane about mn until it

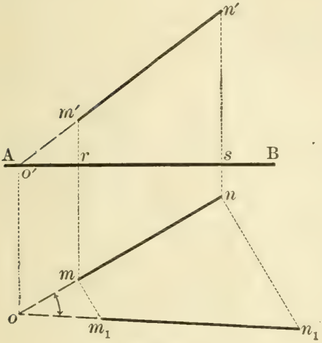


FIG. 18.

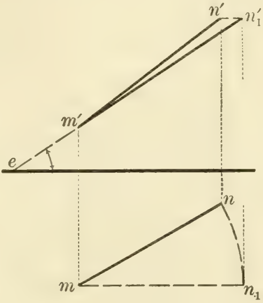


FIG. 19.

coincides with H; the points M and N will fall at m_1 and n_1, at distances from m and n equal to rm' and sn' respectively; join m_1 and n_1, and m_1n_1 will be the required distance.

Since the point o in which the line produced pierces H is in the axis, it remains fixed. The line m_1n_1 produced must then pass through o, and the accuracy of the drawing may thus be verified.

The angle mom_1 is the angle made by the line MN with the horizontal plane. How would you find the angle with the vertical plane?

26. Second method for the same problem.

Analysis. If we revolve the horizontal projecting plane of the line about the projecting perpendicular of either of its points until it becomes parallel to the vertical plane, the line will, in its revolved position, be projected on this plane in its true length (Prop. XIV, Art 14). If we then construct this vertical projection, it will be the required distance.

Construction. Revolve the horizontal projecting plane, Fig. 19, about the perpendicular at m. The point n describes the arc nn_1 until it comes into the line mn_1 parallel to the ground line; n_1 will be the horizontal projection of N in its revolved position. Its vertical projection must be in $n_1n'_1$ perpendicular to the ground line; and since during the revolution the point N remains at the same distance above H, its vertical projection must also be in the line $n'n'_1$ parallel to the ground line (Prop. I, Art. 7), therefore it will be at n'_1.

The point M, being in the axis, remains fixed, and its vertical projection remains at m'; $m'n'_1$ is then the vertical projection of MN in its revolved position, and the true distance.

By examining the drawing, it will be seen that the true distance is the hypotenuse of a right-angled triangle whose base is the horizontal projection of the line, and altitude the difference between the distances of its two extremities from the horizontal plane. Also, that *the angle at the base is equal to the angle made by the line with the horizontal plane.* Also, that the length of the line is always greater than that of its projection, unless it is parallel to the plane of projection.

Ex. 17. Assume lines situated similarly to KL and MN, Fig. 11, and find the true length of each by the first method. Also determine the angle that each line makes with H and with V.

Ex. 18. Assume a point in the first angle, and one in the third. Find the distance between them by the first method and then by the second. Compare results.

Ex. 19. Find the projections of a point X in MN, Fig. 11, at a distance of 1″ from M.

Ex. 20. Assume a line BC, parallel to the profile plane, and find where it pierces H and V.

Ex. 21. Find a point W in the line MN, Fig. 18, equally distant from H and V.

Ex. 22. A rod 2½ ft. long is suspended horizontally by vertical threads 3 ft. long attached to its extreme ends. How far will the rod be raised by turning it through a horizontal angle of 60°?

Ex. 23. Construct the projections of a 1½ in. cube, one "body diagonal" being perpendicular to H, and the H projection of one edge being perpendicular to the ground line.

Ex. 24. Assume a point P in the third angle. It represents a particle acted upon by three forces not in the same plane. Represent these forces by three lines of different lengths, and find the direction and intensity of their resultant.

27. Problem 3. **To assume a straight line lying in a given plane.**

Every straight line of a plane must pierce any other plane to which it is not parallel in the common intersection of the two. Hence (Prop. XXI, Art. 14) if we take a point in each trace and join the two by a straight line, the line will lie wholly in the plane. Or, we may draw the H projection, and at the points where it intersects the ground line and the horizontal trace erect perpendiculars to the ground line; join the point where the first intersects the vertical trace with the point where the second intersects the ground line — this will be the vertical projection of the line.

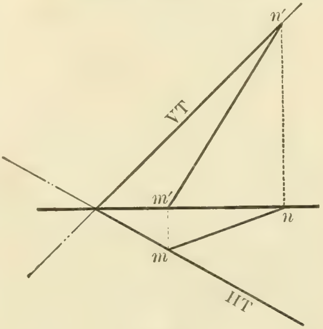


Fig 20.

Thus in Fig. 20, assume *mn* as the H projection of a line lying in the plane T. The point *m* where it intersects the H trace is vertically projected in the ground line at *m'* (Art. 8).

The point n where it intersects the ground line is vertically projected in the V trace at n' (Art. 8). Joining m' and n', we have the V projection of the line.

Second Case. **If the H projection of the line is parallel to the H trace of the plane in which it lies,** its V projection must be parallel to the ground line (Prop. XXII, Art. 14). Hence we have only to produce the H projection to the ground line, find the V projection of that point, and through it draw the V projection of the line parallel to the ground line. See the line MP, Fig. 21.

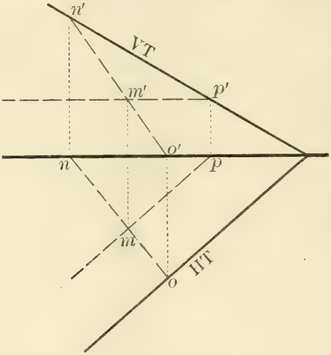


Fig. 21.

Ex. 25. Represent a line CD in the plane R, Fig. 12, assuming the H projection parallel to the ground line.

Ex. 26. Represent a line EF in the plane S, Fig. 12, assuming the V projection parallel to the ground line.

Ex. 27. Represent a line MN in the plane T, Fig. 12, assuming the V projection parallel to the V trace.

28. Problem 4. **Given either projection of a point in an oblique plane, to determine the other projection.**

If through the given projection of the point the corresponding projection of any line of the plane be drawn, and then the other projection of the line be found, the undetermined projection of the point will lie in the corresponding projection of the line (Prop. V, Art. 13), and in a perpendicular to the ground line drawn through the given projection of the point (Prop. XXVI, Art. 16). Thus let m, Fig. 21, be the H projection of a point of the plane T. Through it draw the H projection *no* of any line of the plane. This line is vertically projected in $n'o'$ by the previous problem, and the V projection of the point will be found in this line and also in the perpendicular to the ground line mm'. Or, the point m' might have been found by drawing mp, pp', $p'm'$, mm'.

Ex. 28. Represent a point M in each of the planes R, T, and W, Fig. 12, assuming the H projection first and then finding the V projection.

Ex. 29. Represent a point N in each of the planes S, U, and Y, Fig. 12, assuming the V projection first, and then finding the H projection.

29. PROBLEM 5. **To pass a plane through three given points.**

Let M, N, and P, Fig. 22, be the three points.

Analysis. If we join any two of the points by a straight line, it will lie in the required plane, and pierce the planes of projection in the traces of this plane (Prop. XXI, Art. 14). If we join one of these points with the third point, we shall have a second line of the plane. If we find the points in which these lines pierce the planes of projection, we shall have two points of each trace. The traces, and therefore the plane, will be fully determined.

Construction. Join m and n by the straight line mn; also m' and n' by $m'n'$. MN will be the line joining the first two points. This pierces H at h, and

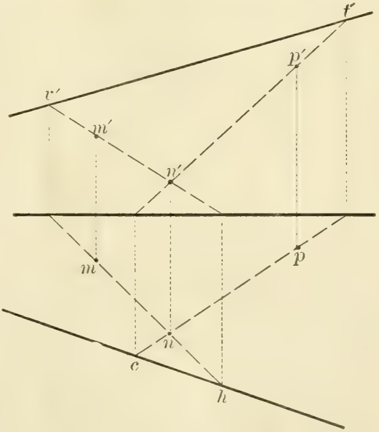


FIG. 22.

V at v', as in Problem 1. Draw also np and $n'p'$; NP will be the second line. It pierces H at c and V at t'. Join h and c by the straight line hc; it is the required horizontal trace. Join v' and t'; $t'v'$ is the vertical trace. Or, produce hc until it meets the ground line, and join this point with either v' or t' for the vertical trace (Prop. VII, Art. 14).

If either MN or NP should be parallel to the ground line, the plane, and consequently its traces, will be parallel to the ground line (Prop. VIII, Art. 14), and it will be necessary to find only one point in each trace.

30. To pass a plane through two straight lines which either intersect or are parallel, we have simply to find the points in which these lines pierce the planes of projection, as in the preceding problem. If the lines do not pierce the planes of projection within the limits of the drawing, then any two points of the lines may be joined by a straight line, and a point in each trace may be determined by finding the points in which this line pierces the planes of projection.

31. To pass a plane through a point and a straight line, join the point with any point of the line by a straight line, and then pass a plane through these lines; or draw through the point a line parallel to the given line, and then pass a plane through the parallels, as above.

Ex. 30. Assume a point A in the second angle, a point B in the third angle, and a point C in the fourth angle, and pass a plane S through them.

Ex. 31. Assume two lines, WX and YZ, each parallel to MN, Fig. 11, and pass through them a plane T.

Ex. 32. Assume two intersecting lines, BC and DE, one parallel to H and oblique to V, the other parallel to V and oblique to H. Pass a plane R through them.

Ex. 33. Assume a line similar to KL, Fig. 11, and a point X in the third angle. Pass a plane T through the point and the line.

Ex. 34. Assume a line MN whose projections do not intersect the ground line within the limits of the drawing. Assume another line OP, parallel to MN, and determine the plane S of the two lines.

Ex. 35. Assume two lines CD and EF each parallel to the ground line, one in the first angle and one in the third angle. Pass a plane T through them.

32. Problem 6. **To pass a plane through a given point parallel to two given straight lines.**

Through the given point draw a line parallel to each of the given lines. The plane of these two lines will be the required plane, since it contains a line parallel to each of the given lines.

33. Problem 7. **To pass a plane through a given straight line parallel to another straight line.**

Through any point of the first line draw a line parallel to the second. Through this auxiliary line and the first pass a plane. It will be the required plane.

Ex. 36. Assume a point C in the second angle, a line MN in the first angle, and a line OP in the third angle. Pass a plane T through the point and parallel to both lines.

Ex. 37. Make the construction for the above problem when MN is parallel to the ground line.

Ex. 38. Assume a line AB in the first angle and a line EF in the fourth angle. Pass a plane S through EF and parallel to AB.

Ex. 39. Make the construction for the above problem when EF is parallel to the ground line.

Ex. 40. Given two lines neither parallel nor intersecting, to pass a plane parallel to both and equally distant from each.

Hint: Draw a straight line from a point in the first line to a point in the second. Through the middle point of this line pass a plane parallel to the two given lines (Art. 32).

34. Problem 8. **Given a line lying in H (or V), and a point in space, to find the position of the point after it has been revolved into H (or V) about the line as an axis.**

Any geometrical magnitude or object is said to be revolved about a straight line as an axis when it is so moved that each of its points describes the circumference of a circle whose plane is perpendicular to the axis, and whose center is in the axis.

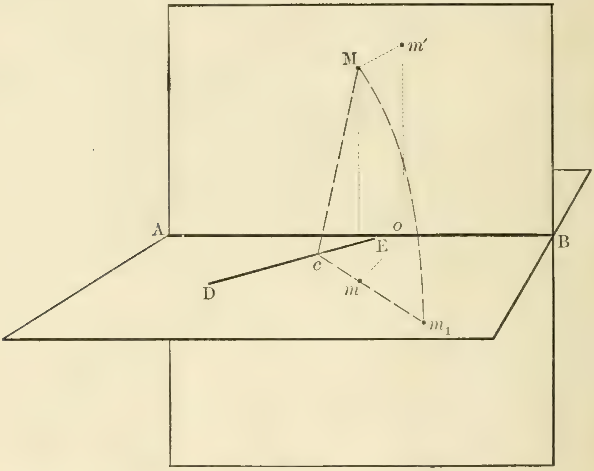


Fig. 23.

If the point M, Figs. 23 and 23 *a*, is revolved about an axis DE, in the horizontal plane, it will describe the circumference of a circle whose center is at *c* and whose radius is M*c*; and since the point must remain in the plane perpendicular to DE, when it reaches the horizontal plane it will be at m_1 in the perpendicular cmm_1, at a distance from *c* equal to M*c*; that is, it will be found *in a straight line passing through its horizontal projection perpendicular to the axis*, and at a distance from the axis equal *to the hypotenuse of a right-angled triangle of which the base* (*mc*) *is the distance from the horizontal projection to the axis, and the altitude* (M*m*) *equal to the distance of the point from the horizontal plane, or equal to the distance* (*m'o*) *of its vertical projection from the ground line.*

Likewise, if a point be revolved about an axis in the vertical plane until it reaches the vertical plane, its revolved position will be found by the same rule, changing the word *horizontal* into *vertical*, and the reverse.

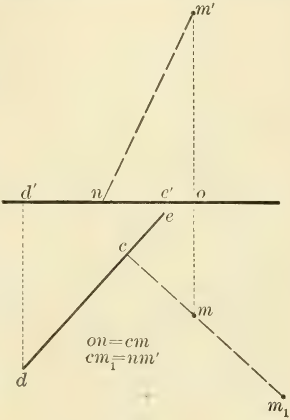


Fig. 23 *a*.

35. Problem 9 *a*. **To find the angle between two straight lines which intersect.**

Let MO and PO, Fig. 24, be the two straight lines, assumed as in Prop. XXVIII, Art. 16.

Analysis. Since the lines intersect, a plane may be passed through them. Revolve this plane about its horizontal trace until it coincides with the horizontal plane, and find the revolved position of the two lines. Since they do not change their relative position, their angle, in this new position, will be the required angle.

Construction. The line MO pierces H at *n*, and the line PO at *p* (Art. 24); *np* is then the horizontal trace of the plane containing the two lines (Art. 30). Revolve this plane about *np* until it coincides with H. The point O falls at o_1 (Art. 34); the distance so_1 being equal to qo', the hypotenuse of a right triangle whose base *bq* is equal to the distance *so*, and whose altitude is the distance of *o'* from the ground line. The points

n and p, being in the axis, remain fixed; o_1n will then be the revolved position of ON, and o_1p of OP; and the angle no_1p will be the required angle.

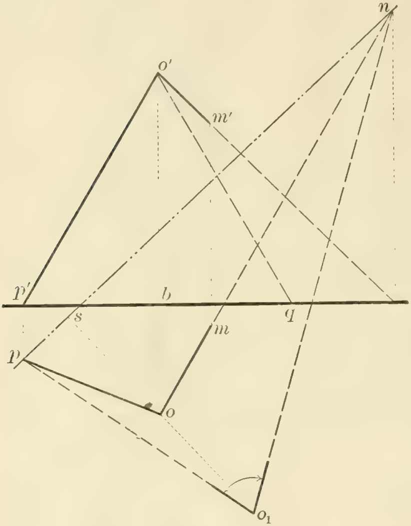


Fig. 24.

Note. The supplement of the angle no_1p may also be regarded as the angle between the lines; but unless otherwise specified the *lesser* of the two will be regarded as the one required.

36. Problem 9 *b*. To find the angle between two intersecting lines, when one of them is parallel to either H or V.

Let it be required to find the angle between DE and EF, Fig. 25, the latter being parallel to H (Prop. XIV, Art. 14).

In this case it is unnecessary to pass a plane through the two lines, since the line EF may be used as an axis about which ED may be revolved until it is parallel to the H plane. The angle will then be projected upon H in its true magnitude (Prop. XXIII, Art. 14). As a result of this revolution the horizontal projection of any point in the revolved line, as d, will move in a perpendicular to ef a distance from it equal to od_1, the hypotenuse of a right triangle whose base is equal to do and whose altitude is the distance of d' from the V projection of the axis.

During this revolution the vertex E does not move. Hence *ed* takes the new position ed_1, and the angle fed_1 is the angle required.

37. PROBLEM 10. **To find the shortest distance from a given point to a given straight line.**

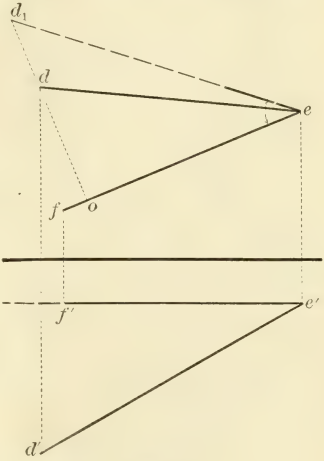


FIG. 25.

Analysis. The required distance is the length of a perpendicular from the point to the line. If through the given point and the line we pass a plane, and revolve this plane about either trace until it coincides with the corresponding plane of projection, the line and point will not change their relative positions; hence, if through the revolved position of the point we draw a perpendicular to the revolved position of the line, it will be the required distance.

Let the student make the construction in accordance with this analysis.

EX. 41. Assume three points in space, A, B, and C. Join them by lines, forming a triangle. Determine the true size and shape of this triangle by the method of Problems 8 and 9 *a*.

EX. 42. Assume the projections of a parallelogram in the third angle (Prop. XIX, Art. 14), and determine its true size and shape.

EX. 43. The H trace of a plane T makes an angle of 60° with the ground line. The angle between the two traces *in space* is 75°. What angle does the V trace make with the ground line?

Ex. 44. Assume a, b, c, d, e, the H projections of the five vertices of a plane pentagon, and a', b', c', the V projections of three of them. Find d', e', and the true figure, without constructing the plane of the pentagon.

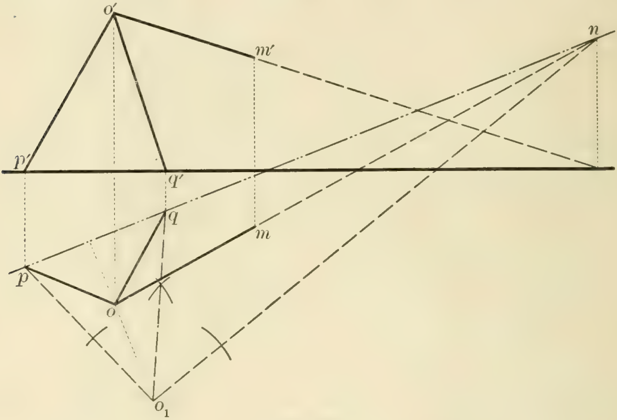


FIG. 26.

38. PROBLEM 11 *a*. **To construct the projections of a line bisecting the angle between two given lines.**

Let OM and OP, Fig. 26, be the two lines, intersecting in the point O.

Analysis. If the plane of the two lines be revolved about its horizontal trace into H, the angle will then be shown in its true size (Art. 35) and may be bisected by a straight line. If the plane be then revolved to its primitive position, and the true position of one point of the bisector be determined, and joined with the vertex of the given angle, we shall have the required line.

Construction. Let po_1n be the revolved position of the angle. Bisect it by o_1q, which will be the horizontal projection of the bisector in its revolved position. Revolving the plane to its primitive position, the point q of the bisector remains fixed, while o_1 returns to o. Joining o and q, we have the H projection of the bisector.

Since the point Q lies in H, its vertical projection must be in the ground line at q'. Joining q' with o', we have the vertical projection of the required bisector.

39. Problem 11 *b*. **To construct the projections of a line bisecting the angle between two given lines, one of which is parallel to H or V.**

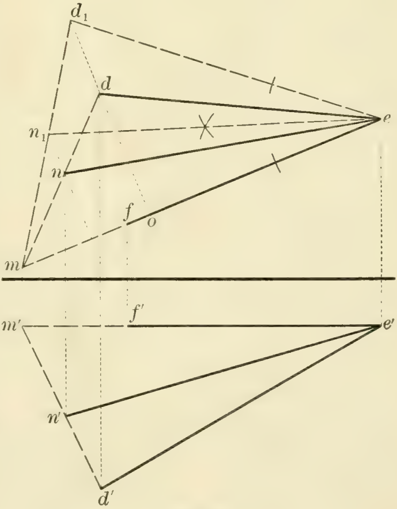


Fig. 27.

In this case the true magnitude of the angle is found as in Fig. 25, the axis of revolution being the given line parallel to one of the planes of projection.

Let EF and ED, Fig. 27, be the two given lines, EF being parallel to H. Revolving ED about EF as an axis until it is parallel to H, as in Fig. 25, the true angle between the lines is found to be equal to fed_1. The line en_1 is the H projection of the bisector in its revolved position. To bring it back to its primitive position, join d_1 with any convenient point in ef, as m. The bisector cuts this auxiliary line in n_1. When ED is revolved to its primitive position, d_1 returns to d, m remains fixed, d_1m takes the position dm, n_1 moves along a perpendicular to the axis (Art. 34) to the position n. en is then the horizontal projection of the bisector. The auxiliary line MD is vertically projected in $m'd'$, N is vertically projected in n' (Prop. V, Art. 13), and $e'n'$ is the vertical projection of the bisector.

Note. The use of an auxiliary line (of which the above is an example) joining the revolved position of a known point with a point in the axis of revolution, is applicable to many problems where it is desired to bring points from their revolved to their primitive positions.

Ex. 45. Assume two intersecting lines, AB and AC, one of which is parallel to the ground line. Find the angle between them, and the projections of the bisector AN.

Ex. 46. Assume two intersecting lines, MN and MP, one of which is perpendicular to V. Find the angle between them and the projections of the bisector.

Ex. 47. Assume two intersecting lines similar to PO and MO, Fig. 26, and determine the projections of the bisector of the supplementary angle formed by producing MO beyond O.

Ex. 48. Assume a line MN and a point P outside the line. Construct the projections of a line PC making an angle of 60° with MN.

Ex. 49. Assume a plane S, and a line AB lying in that plane. Through B pass a line BC lying in the plane and making an angle of 60° with AB.

Ex. 50. Assume a line in the third angle, and a point C outside the line. Construct the projections of a regular hexagon whose center is at the point C and one of whose sides lies in the assumed line.

40. PROBLEM 12 *a*. **To find the intersection of two planes.**

Let T and S, Fig. 28, be the two planes.

Analysis. Since the line of intersection is a straight line contained in each plane, it must pierce the horizontal plane in the horizontal trace of each plane (Prop. XXI, Art. 14); that is, *at the intersection of the two traces.* For the same reason, it must pierce the vertical plane at the intersection of the vertical traces. If these two points be joined by a straight line, it will be the required intersection.

Construction. The required line pierces H at *o* and V at *p'*: *o* is its own horizontal projection, and *p'* is horizontally projected at *p*; hence *po* is the horizontal projection of the required line; *o* is vertically projected at *o'*; *p'* is its own vertical projection; and *o'p'* is the vertical projection of the required line.

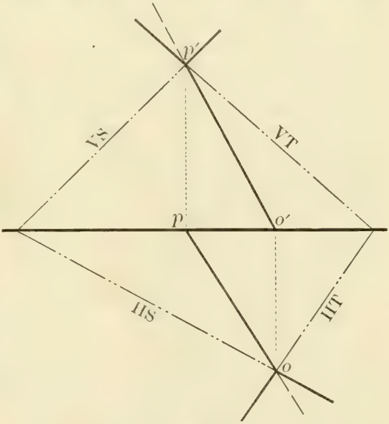


FIG. 28.

41. PROBLEM 12 *b*. **To find the intersection of two planes when either the horizontal or the vertical traces do not intersect within the limits of the drawing.** Let T and S, Fig. 29, be the planes; HT and HS not intersecting within the limits of the drawing.

Analysis. If we pass any plane parallel to the vertical plane, it will intersect each of the given planes in a line parallel to its vertical trace, and these two lines will intersect in a point of the required intersection. A second point may be determined in the same way, and the straight line joining these two points will be the required line.

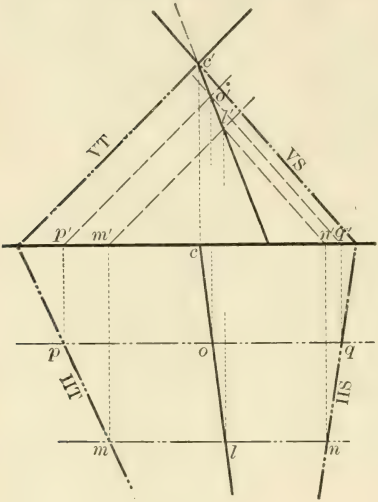


Fig. 29.

Construction. Draw *mn* parallel to the ground line; it will be the H trace of an auxiliary plane. It intersects the two given planes in lines which pierce H at *m* and *n*, and are vertically projected in *m'l'* and *n'l'*; *l'* is the vertical, and *l* the horizontal projection of their intersection. Draw *pq* also parallel to the ground line, and thus determine O. LO is the required line.

Or, since the point C in which the vertical traces intersect is a point of the required line, it may be joined directly to the point L; the second auxiliary plane being unnecessary unless *neither* pair of traces intersect within the limits of the drawing.

Ex. 51. Assume planes similar to R and T, Fig. 12, and find their intersection.

Ex. 52. Assume planes similar to R and S, Fig. 12, but with all four traces intersecting the ground line at the same point. Find the line of intersection by the method of Problem 12 *b*.

Ex. 53. Find the intersection of two planes, both parallel to the ground line. (It will here be convenient to make use of the profile plane of projection.)

Ex. 54. Assume two oblique planes, S and T, and a point M outside of both. Through M pass a line parallel to both planes.

Ex. 55. Given two intersecting planes (oblique) and a point in each, to find the shortest path in these planes between the two points. Analyze and construct.

42. Problem 13 *a*. **To find the point in which a given straight line pierces a given plane.**

Let MN, Fig. 30, be the given line, and T the given plane.

Analysis. If through the line any plane be passed, it will intersect the given plane in a straight line, which must contain the required point. This point must also be on the given line; hence it will be at the intersection of the two lines.

Construction. Let the auxiliary plane be the horizontal projecting plane of the line; *np* is its horizontal and *pp'* its vertical trace (Prop. X, Art. 14). It intersects T in a straight line, which pierces H at *o* and V at *p'*, the vertical projection being *o'p'* (Art. 40). The point *m'*, in which *o'p'* intersects *m'n'*, is the vertical projection of the required point; and *m* is its horizontal projection. The accuracy of the drawing may be verified by using the vertical projecting plane of MN, as an auxiliary plane, and determining *m* directly, as represented in the figure.

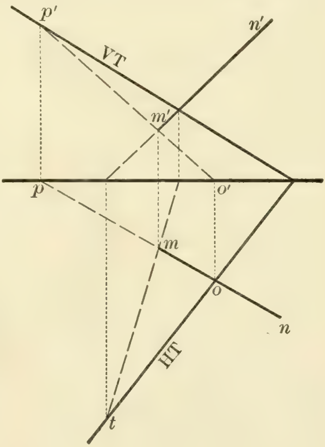


Fig. 30.

Rule. *Produce the* H *projection of the line to the* H *trace and to the ground line. At these points erect perpendiculars to the* *ground line. Draw a straight line from the point where the first intersects the ground line to the point where the second intersects the* V *trace. The point where this line cuts the* V *projection of the given line is the* V *projection of the piercing point. This rule will also apply when* H *is changed to* V *and* V *to* H.

Ex. 56. Assume a plane like S, Fig. 12, and a line MN in the second angle, perpendicular to the plane (Prop. XXIV, Art. 14). Find the point C where the line pierces the plane.

Ex. 57. Assume a plane like R, Fig. 12, and a line EB parallel to the ground line. Find the point P where the line pierces the plane.

Ex. 58. Assume a plane T parallel to the ground line, and a line EF parallel to the profile plane. Find where the line pierces the plane.

Ex. 59. Assume a plane like T, Fig. 12, and a point C in the second angle. Find the shortest distance from the point to the plane. See Prop. XXIV (Art. 14), Art. 42, and Art. 25.

Ex. 60. Assume a point P, a line ED, and a plane S. The point P is a source of light. Find the shadow cast by ED upon the plane S.

Ex. 61. Assume a plane like R, Fig. 12, and a line BC in the third angle whose H projection is parallel to HR and whose V projection is parallel to VR. Is this line parallel to the plane? Project the line upon the plane and show the H and V projections of this projection. Remember that the projection of a point upon *any* plane is the foot of a perpendicular dropped from the point to the plane.

Ex. 62. Through a given line MN to pass a plane T perpendicular to a given plane S. Analyze and construct.

Ex. 63. Assume a point P in the third angle, a line MN in the second angle, and a plane T. Pass a plane S through P, parallel to MN and perpendicular to the plane T.

Ex. 64. Assume a plane T and a line MN oblique to it. Pass a line OP lying in the plane and perpendicular to MN. Analyze and construct.

Ex. 65. Assume two lines, AB and CD, not in the same plane, and a point P not lying in either line. Through the point pass a third line PQ touching both AB and CD.

Ex. 66. A ray of light emanating from a given point strikes a given oblique plane. Find the points 1, 2, and 3 where it is successively reflected from the oblique plane and the two planes of projection.

Ex. 67. Assume a plane like R, Fig. 12, and an oblique line EF lying in that plane. Construct the three projections of a cube resting in R, one edge of the lower face being EF.

Ex. 68. Given three oblique lines, AB, CD, EF, no two of which are in the same plane, to construct the projections of a line XZ parallel to AB, and touching CD and EF. Analyze and construct.

43. Problem 13 *b*. **To find the point in which a given straight line pierces a plane, when the plane is given by any two of its straight lines.**

Analysis. If we find the points in which these two lines pierce either projecting plane of the given line, and join these points by a straight line, this will intersect the given line in the required point.

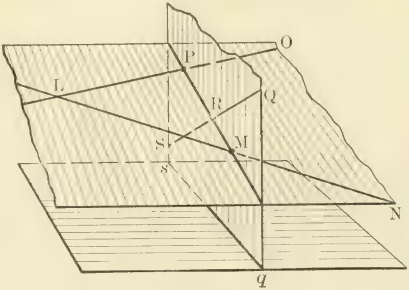


Fig. 31.

Construction. Let LN and LO, Figs. 31 and 31 *a*, be the lines of the given plane, intersecting at L, and QS be the given line. The line LN pierces the horizontal projecting plane of QS at a point of which m is the horizontal, and m' the vertical projection. LO pierces the same plane at P, and $p'm'$ is the vertical projection of the line joining these two points. This intersects $q's'$ at r', which is the vertical projection of the required point, r being its horizontal projection.

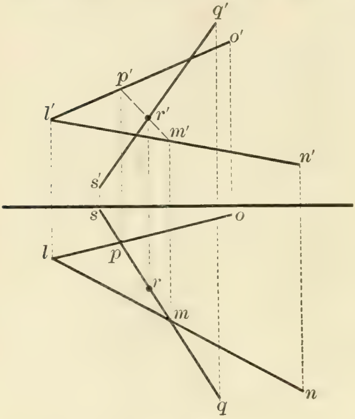


Fig. 31 a.

Ex. 69. Assume two parallel lines, AB and CD, in the third angle, and a line XZ. Find the point P where XZ pierces the plane of AB and CD.

Ex. 70. Assume three points in space, A, B, and C, and find the point X in which a line MN pierces the plane of the three points, without constructing the traces of the plane.

44. Problem 14. **Through a given point to pass a plane perpendicular to a given straight line.**

Let M, Fig. 32, be the given point, and NO the given line.

Analysis. Since the plane is to be perpendicular to the line, its traces must be respectively perpendicular to the projections of the line (Prop. XXIV, Art. 14). We thus know the direction of the traces. Through the point draw a line parallel to the horizontal trace; it will be a line of the required plane, and will pierce the vertical plane in a point of the vertical trace. Through this point draw a straight line perpendicular to the vertical projection of the line; it will be the vertical trace of the required plane. Through the point in which this trace intersects the ground line draw a straight line perpendicular to the horizontal projection of the line; it will be the horizontal trace.

Construction. Through *m* draw *mp* perpendicular to *no*; it will be the horizontal projection of a line through M, parallel to the horizontal trace; and since this line is parallel to H, its vertical projection will be $m'p'$ parallel to the ground line. This line pierces V at p' (Art. 24). Draw p'T perpendicular to $n'o'$, and HT perpendicular to *no*; HT–VT will be the required plane. Or, through M draw a line parallel to the vertical trace. It pierces H in a point of the horizontal trace, and the accuracy of the drawing may thus be tested.

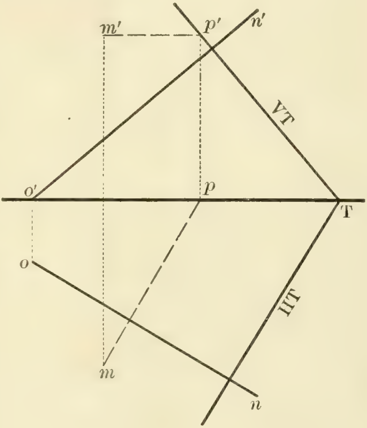


FIG. 32.

Ex. 71. Pass a plane S through a given point M parallel to a given plane T. Analyze and construct.

Ex. 72. Pass a plane S parallel to a given plane R and at a given distance from it. Analyze and construct.

Ex. 73. Given three points, A, B, C, not in the same straight line, to find a point in H equally distant from them. Analyze and construct.

Ex. 74. Assume an oblique plane R and an oblique line MN lying in the plane. Pass a plane T through MN and making an angle of 45° with R. Analyze and construct.

Ex. 75. Assume a plane like R, Fig. 12, and two oblique lines, AB and CD. Construct the projections of a third line MX, parallel to the plane R and touching AB and CD. Analyze and construct.

Ex. 76. Assume a point M, a line EF, and a plane S. Through M pass a line MX, touching EF and parallel to S. Analyze and construct.

45. Problem 15. **To construct the projections of a circle lying in a given plane, its diameter and the position of its center being known.**

Let the plane T, Fig. 33, be the plane of the circle, and C (assumed as in Art. 28), its center.

Analysis. The projection of any circle upon a plane to which it is neither parallel nor perpendicular will be an *ellipse* whose major axis is the projection of that diameter which is parallel to the trace of the plane of the circle on the plane of projection, and whose minor axis is the projection of the diameter perpendicular to that trace. The length and direction of the major axis being known, it can be drawn at once (Props. XIV and XXII, Art. 14). The extremities of the minor axis may be found by revolving the circle about its trace into the plane of projection, constructing the revolved position of the two axes, and returning them to their primitive positions by the principle noted in Art. 39.

Construction. Revolving the plane T about its horizontal trace into H, the center C falls at c_1, and the circle takes the position $a_1e_1d_1g_1$. The diameter a_1d_1, parallel to HT, is projected upon H in its true length when in its proper position (Prop. XIV, Art. 14). Its H projection is therefore *ad*. The H projection of the diameter e_1g_1 perpendicular to HT, becomes the minor axis of the ellipse. To find its extremities, draw a_1g_1k and e_1d_1m. The points a_1 and g_1 revolve back to *a* and *g* while *k* remains fixed; also points e_1 and d_1 revolve back to *e* and *d*, while *m* remains fixed. The lines then take the

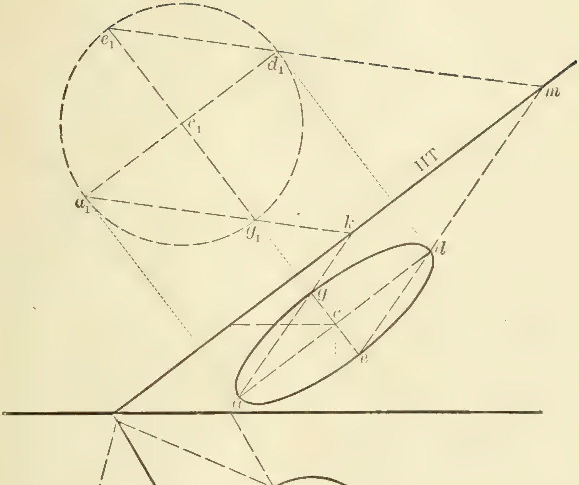


Fig. 33.

positions *kga* and *mde* respectively, and the line *ge* is then the H projection of the diameter GE, and forms the minor axis of the ellipse into which the circle is horizontally projected. The vertical projection may be found by a repetition of this process with respect to the vertical trace, as shown in the figure.

Points on the elliptical curves may be found by the "*trammel method*" thus: Take a stiff strip of paper with a perfectly straight edge *xz* and a sharp corner *x*. Lay off *xy* equal to half the minor axis, and *xz* equal to half the major axis. Then place the strip in various positions, always keeping *y* upon the major and *z* upon the minor axis, and the corner *x* will give points on the ellipse. Having found a sufficient number of them, a smooth curve may be drawn by means of a draftsman's "irregular curve."

Ex. 77. Given a point P and a line MN, to construct the projections of a circle whose center lies in MN and whose circumference passes through P. Analyze and construct.

Ex. 78. Assume a point A in the first angle, a point B in the second angle, and a point C in the third angle. Join them by straight lines forming a triangle ABC. Construct the projections of the inscribed circle.

Ex. 79. Given two intersecting lines and the diameter of a circle tangent to both, to construct the projections of the circle. Analyze and construct.

Ex. 80. A given point P is revolved about a given line MN as an axis through an arc of 90°. Show the projections of its new position and of its path. Analyze and construct.

Ex. 81. Assume a point P, and an oblique line MN in the third angle. The point is revolved about the line as an axis, and pierces H and V in the points X and Y respectively. Locate these points.

Ex. 82. Assume a plane like R, Fig 12, and a point S outside the plane. With S as a vertex construct the three projections of a right circular cone with base in R and elements making an angle of $22\frac{1}{2}°$ with the axis.

Note. — Exercises 82, 92, and 93 may, if the instructor prefers, be postponed till after the treatment of the generation and classification of surfaces.

46. PROBLEM 16. **To find the angle which a given straight line makes with a given plane.**

Let MN, Fig. 34, be the given line, and T the given plane.

Analysis. The angle made by the line with the plane is the same as that made by the line with its projection on the plane. Hence, if through any point of the line a perpendicular be drawn to the plane, the foot of this perpendicular will be one point of the projection. If this point be joined with the point in which the given line pierces the plane, we shall have the projection of the line on the plane. This projection, the perpendicular, and a portion of the given line form a right-angled triangle, of which the projection is the base, and the angle at the base is the required angle. But the angle at the vertex, that is, *the angle between the perpendicular and the given line*, is the complement of the required angle; hence, if we find the latter angle and subtract it from a right angle, we shall have the required angle.

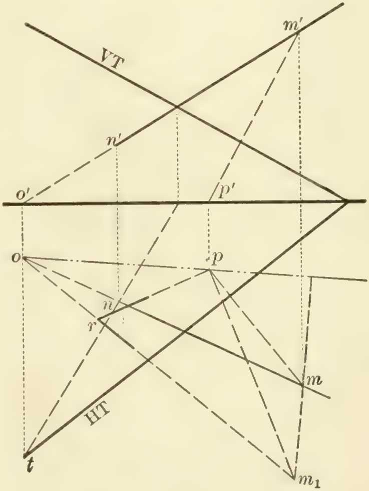


FIG. 34.

Construction. Through M draw the perpendicular MP to the plane T (Prop. XXIV, Art. 14). It pierces H in *p*. The given line pierces H in *o*, and *op* is the horizontal trace of the plane of the two lines (Art. 30). Revolve this plane about *op*, and determine their angle pm_1o, as in Art. 35. Its complement, prm_1, is equal to the required angle.

Ex. 83. Let the above problem be constructed when the plane is parallel to the ground line.

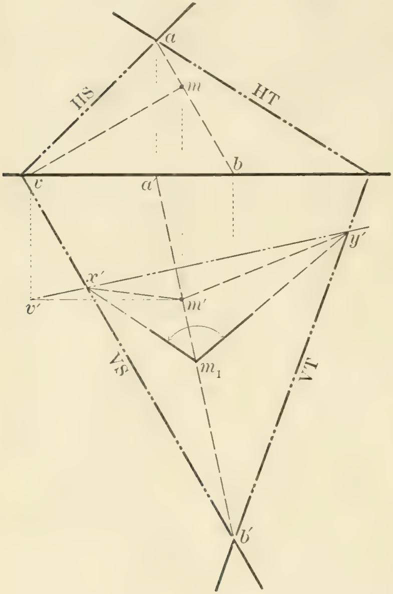


FIG. 35.

47. PROBLEM 17 *a.* **To find the angle between two given planes.**

Analysis. If we pass a plane perpendicular to the intersection of the two planes, it will be perpendicular to both, and cut from each a straight line perpendicular to this intersection at a common point. The angle between these lines will be the measure of the required angle.

Construction. Let the planes S and T, Fig. 35, be the given planes. Their intersection is AB (Art. 40). M is any assumed point in AB. A plane is passed through M perpendicular to AB (Art. 44), and *v'y'*, perpendicular to *a'b'*, is its vertical trace. This plane cuts the planes S and T in lines, of which $m'x'$ and $m'y'$ are the vertical projections. The angle between the lines MX and MY is found by revolving the auxiliary plane about $x'y'$ into coincidence with V (Art. 35). m_1 is the revolved position of the vertex, and the angle $x'm_1y'$ is the required angle.

48. Second method for the same problem.

Analysis. If from any point in space two lines be passed perpendicular respectively to the two given planes (Prop. XXIV, Art. 14), the angle included between the lines will be equal to the angle between the planes, or its supplement (Art. 35). Let the student make the construction in accordance with this analysis.

49. PROBLEM 17 *b*. **To find the angle between a given plane and either plane of projection,** as the horizontal, we simply pass a plane perpendicular to their intersection (in this case the horizontal trace), as in Fig. 36. This plane cuts *on* from H, and ON from the plane T, and the angle non_1, found by revolving the auxiliary plane about *on* (Art. 35), will be the required angle $nn_1 = nn'$.

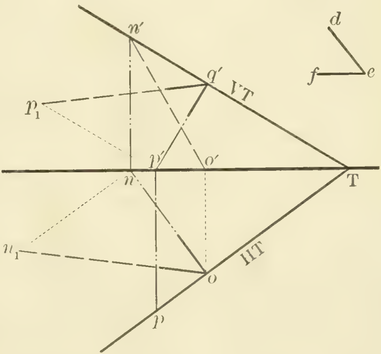


FIG. 36.

In the same way the angle $p'q'p_1$, between the given plane and vertical plane, may be found.

Ex. 84. Find the angle between two planes, both parallel to the ground line.

Ex. 85. Assume two intersecting planes as in Fig. 28, and find the traces of the plane that bisects the dihedral angle between them.

Ex. 86. Given two planes and a straight line, to find a point in the line equally distant from the two planes.

Ex. 87. Assume four points A, B, C, and D, in the third angle. Let them be taken as the vertices of a tetrahedron. Find the angle between two of the faces without drawing the traces of their planes. Analyze and construct.

50. Problem 18. **Either trace of a plane being given, and the angle which the plane makes with the corresponding plane of projection, to construct the other trace.**

Let HT, Fig. 36, be the horizontal trace of the plane, and *def* the angle which the plane makes with the horizontal plane.

Analysis. If a straight line be drawn through any point of the given trace, perpendicular to it, it will be the horizontal trace of a plane perpendicular to the given trace, and if at the same point a line be drawn, making with this line an angle equal to the given angle, this will be the revolved position of a line cut from the required plane by this perpendicular plane (Art. 49). If this line be revolved to its true position, and the point in which it pierces the vertical plane be found, this will be a point of the required vertical trace. If this point be joined with the point where the horizontal trace intersects the ground line, we shall have the vertical trace.

Construction. Through o draw on perpendicular to HT; also on_1, making the angle $non_1 = def$; on_1 will be the revolved position of a line of the required plane. When this line is revolved to its true position, it pierces V at n', nn' being equal to nn_1, and n'T is the required trace.

If the given trace does not intersect the ground line within the limits of the drawing, the same construction may be made at a second point of the trace, and thus another point of the vertical trace be determined.

Ex. 88. Assume the H trace of a plane T parallel to the ground line. The plane makes an angle of 60° with H. Determine the V trace.

Ex. 89. Assume the H trace of a plane, inclined 30° to the ground line. The plane makes an angle of 45° with V. Determine the V trace. Analyze and construct.

Ex. 90. Given the H trace of a plane T, the projections of a point P, and the distance of the point from the plane, to construct the V trace. Analyze and construct. How many solutions are there?

Ex. 91. Construct the three projections of a 2″ cube, one edge in H making an angle of 30° with the ground line, and one face inclined 30° to H on side *away* from ground line.

Ex. 92. Given a point G, and the H trace of a plane S in which the point lies. Find the V trace, and then construct the projections of a sphere of 2″ diameter tangent to the plane at the point G.

51. PROBLEM 19. **Through a given point to pass a line making given angles with the planes of projection.**

Let it be required to pass a line through C, Fig. 37, making an angle of 30° with H and an angle of 45° with V. If we conceive the line to be of definite length, say 2″, and to be revolved about a vertical line through C until parallel to V, the vertical projection of the line will then be the true length of the line (Art. 26), and its angle with the ground line will be the angle that the line makes with H. Let (cd_1, $c'd'_1$) be the line in this position. If the line is now revolved back to its original position, the H projection of D will take successive positions in the arc d_1x, and the V projection will take successive positions in the line d'_1d' parallel to the ground line.

Now conceive the line to be revolved about an axis through

C perpendicular to the *vertical* plane until it is parallel to H. The horizontal projection of the line will then be the true length of the line, and its angle with the ground line will be the angle that the line makes with V. $(cd_2, c'd'_2)$ is the line in this position. When the line is revolved back to its original position, the horizontal projection of D will take successive positions in a line through d_2 parallel to the ground line. Where this line intersects the arc d_1x must be the proper position for d. Also the vertical projection of D will, during this revolution, take successive positions in the arc d'_2y', and must therefore be at d', the intersection between this arc and the line d'_1d'.

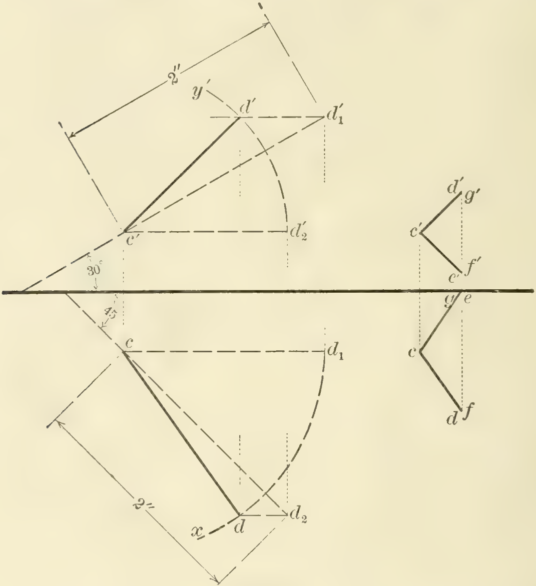


Fig. 37.

There are four possible solutions to this problem, shown in Fig. 37 at the right. If a second line be drawn through c' so as to make the same angle with the ground line as the first, and similarly a second line be drawn through c, we may letter these four lines in such a way that they will represent the projections of four different lines, CD, CE, CF, and CG, all making an angle of 30° with H and an angle of 45° with V.

52. PROBLEM 20. **Through a given point to pass a plane making given angles with the planes of projection.**

Analysis. If a line is perpendicular to a plane, the angles between the line and the planes of projection will be the *complements* of the angles between the plane and the planes of projection. Hence if we construct a line making with H and V the complements of the required angles, a plane through the given point perpendicular to this line will make the required angles with H and V. If, for example, it is required to pass a plane making an angle of 40° with H and an angle of 70° with V, we first construct a line making an angle of 90° – 40° with H and an angle of 90° – 70° with V; then we pass through the given point a plane perpendicular to this line.

Note. The sum of the angles made by a *plane* with H and V *cannot be less than* 90°; and the sum of the angles made by a *straight line* with H and V *cannot exceed* 90°.

Ex. 93. Given a point in the third angle as the center of a pulley turning on an inclined shaft making given angles with H and V. Given also the diameter of the pulley, the diameter of the shaft, and the thickness of the pulley, to construct its two projections.

Ex. 94. Construct the three projections of a regular hexagonal prism $3\frac{1}{2}''$ high, each side of whose base is $1''$, the base making an angle of 30° with H and 75° with V; one edge of the base being parallel to H; the center, C, being $\frac{3}{4}''$ from H and $1\frac{1}{2}''$ from V.

53. PROBLEM 21. **To find the shortest line that can be drawn, terminating in two straight lines, not in the same plane.**

Let MN, Fig. 38, and OP be the two straight lines.

Analysis. The required line is manifestly a straight line, perpendicular to both of the given lines. If through one of the lines we pass a plane parallel to the other, and then project this second line on this plane, this projection will be parallel to the line itself (Prop. XIV, Art. 14), and therefore not parallel to the first line. It will then intersect the first line in a point. If, at this point, we erect a perpendicular to the plane, it will

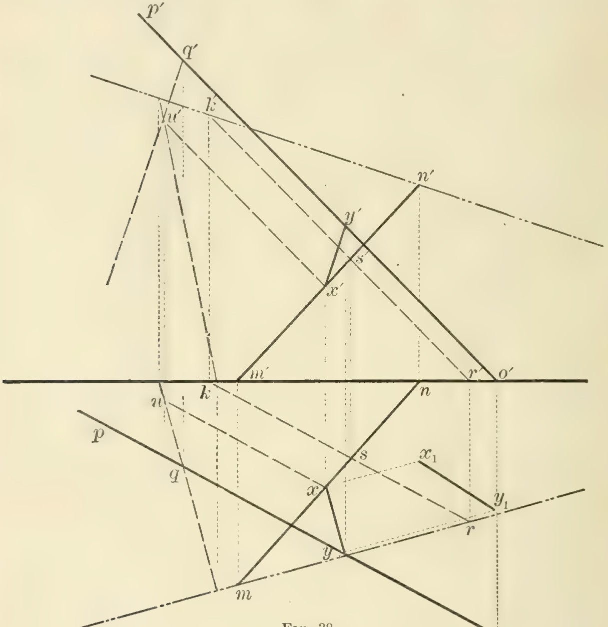


FIG. 38.

be contained in the projecting plane of the second line, be perpendicular to both lines, and intersect them both. That portion included between them is the required line.

Construction. Through MN pass a plane parallel to OP (Art. 33). *mr* is its horizontal, and *k'n'* its vertical trace. Through any point of OP, as Q, draw QU perpendicular to this plane. It pierces the plane at U (Art. 42); and this is one point of the projection of OP on the parallel plane. Through U draw UX parallel to OP; it will be the projection of OP on the plane. It intersects MN in X, which is the point through which the required line is to be drawn, and XY, perpendicular to the plane, is the required line, the true length of which is x_1y_1 (Art. 25).

Ex. 95. Assume two lines, BC and MN, one in the third angle and parallel to the ground line, the other in the second angle and oblique. How can we tell whether they intersect? Find the common perpendicular, XY, to these lines.

Ex. 96. Assume two lines, one of which is perpendicular to H, and find the shortest line between them.

Classification of Lines

54. *Every line may be regarded as generated by the continued motion of a point.* If the generating point be taken in any position on the line, and then be moved to its next position, these two points may be regarded as forming an *infinitely small straight line* or *elementary line.* The two points are *consecutive points*, or points having no distance between them, and may practically be considered as one point.

The line may thus be regarded as made up of an infinite number of infinitely small elements, each element indicating the direction of the motion of the point while generating that part of the line.

55. The law which directs the motion of the generating point determines the nature and class of the line.

If the point moves always in the same direction, that is, so that *the elements of the line are all in the same direction*, the line generated is a *straight line.*

If the point moves so as continually to change its direction from point to point, the line generated is a *curved line*, or *curve.*

If *all the elements of a curve are in the same plane*, the curve is *of single curvature.*

If *no three consecutive elements*, that is, if no four consecutive points, *are in the same plane*, the curve is *of double curvature.*

We thus have **three general classes of lines**:

(1) **Straight** or **right lines**: all of whose points lie in the same direction.

(2) **Curves of single curvature**, or **plane curves**: all of whose points lie in the same plane.

(3) **Curves of double curvature**, or **space curves**: no four consecutive points of which lie in the same plane.

Projection of Curves

56. If all the points of a curve be projected upon the horizontal plane, and these projections be joined by a line, this line is the *horizontal projection* of the curve.

Likewise, if the vertical projections of all the points of a curve be joined by a line, it will be the *vertical projection* of the curve.

57. The two projections of a curve being given, the curve will in general be completely determined. For in the same perpendicular to the ground line two points, one on each projection, may be assumed, and the corresponding point of the curve determined, as in Art. 8. Thus m and m', Fig. 39, being assumed in a perpendicular to the ground line, M will be a point of the curve, and in the same way every point of the curve may in general be determined.

58. If the plane of a curve of single curvature is perpendicular to either plane of projection, the projection of the curve on that plane will be a straight line, and all of its points will be projected into the trace of the plane on this plane of projection.

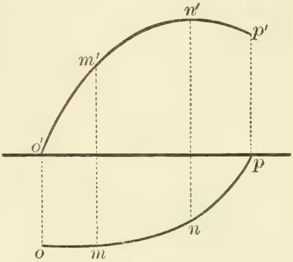


Fig. 39.

If the plane of the curve be perpendicular to the ground line, both projections will be straight lines, perpendicular to the ground line, and the curve will be undetermined, unless the projections of a sufficient number of its points are determined.

If the plane of the curve be parallel to either plane of projection, its projection on that plane will be equal to itself, since each element of the curve will be projected into an equal element (Prop. XIV, Art. 14). Its projection on the other plane will be a straight line parallel to the ground line.

The projection of a space curve can in no case be a straight line.

59. The points in which a curve pierces either plane of projection can be found by the same rule as in Art. 24. Thus *o*, Fig. 39, is the point in which the curve MN pierces H, and p' the point in which it pierces V.

Tangents and Normals to Lines

60. Tangent line. If a straight line be drawn through any point of a curve, as M, Fig. 40, intersecting it in another point, as M′, and then the second point be moved along the curve towards M, until it coincides with it, the line containing both points during the motion will become *tangent to the curve* at M, which is the *point of contact.*

When the point M′ becomes consecutive with M, the line thus containing the element of the curve at M (Art. 54) may, for all practical purposes, be regarded as the tangent; hence we say that *a straight line is tangent to another line when it contains two consecutive points of that line.*

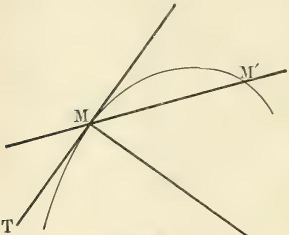


FIG. 40.

Two curves are tangent to each other when they contain two consecutive points, *or have, at a common point, a common tangent.*

If a straight line is tangent to a plane curve, it will be contained in the plane of the curve. For it passes through two points in that plane, viz. the two consecutive points of the curve.

Also, if a straight line is tangent to another straight line, it will coincide with it, as the two lines have two points in common.

The expression "a tangent to a curve" or "a tangent" will hereafter be understood to mean a rectilinear tangent unless otherwise mentioned.

61. Projections of tangents. *If two lines are tangent in space, their projections on the same plane will be tangent to each other.* For the projections of the two consecutive points common to the two lines will also be consecutive points common to the projections of both lines (Art. 56).

The converse of this is not necessarily true. But if both the horizontal and vertical projections are tangent at points which are the projections of a common point of the two lines (Prop. XXVI, Art. 16), the lines will be tangent in space; for the projecting perpendiculars at the common consecutive points will intersect in two consecutive points common to the two lines.

62. Normal. If a straight line be drawn perpendicular to a tangent at its point of contact, as MO, Fig. 40, it is a *normal to the curve.* As an infinite number of perpendiculars can be thus drawn, all in a plane perpendicular to MT at M, there will be an infinite number of normals at the same point.

If the curve be a plane curve, the term "normal" will be understood to mean that normal which is in the plane of the curve unless otherwise mentioned.

63. Rectification. If we conceive a curve to be rolled on its tangent at any point until each of its elements in succession comes into this tangent, the curve is said to be *rectified;* that is, a straight line equal to it in length has been found. Since the tangent to a curve at a point contains the element of the curve, the angle which the curve at this point makes with any line or plane will be the same as that made by the tangent.

64. To rectify a circular arc. Let AB, Fig. 41, be the arc, and AD a tangent to the circle at A. Draw the chord BA, and produce it to E so that AE = $\frac{1}{2}$ BA. With E as center and EB as radius, strike the arc BF. Then AF will be equal to the arc AB, *nearly.* This method is sufficiently accurate only when the angle BCA is less than 60°. For arcs subtending angles greater than 60° the method may be applied to some aliquot part of the arc, and the result spaced off the proper number of times with the dividers. Thus the arc AG is equal to 2 AF or AH.

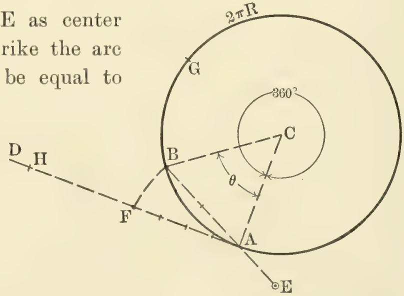


FIG. 41.

If the value of the angle θ is known in degrees, the length of the arc should be *calculated* by means of the relation

$$\frac{\text{Arc AB}}{2\,\pi\text{R}} = \frac{\theta^\circ}{360^\circ},$$

or

$$\text{Arc AB} = 0.01748\,\text{R}\theta^\circ.$$

This calculated result may then be laid off with the draftsman's scale.

65. To lay off a straight line upon a circular arc. Let it be required to lay off the distance AF, Fig. 41, upon the circular arc. Divide AF into four equal parts. With the division nearest to A as center, strike an arc through F intersecting the circle in B. The arc AB will then be equal to AF, *nearly*. This method is not accurate when the subtended angle θ is greater than 60°.

66. To lay off any line, curved or straight, upon any other line. Let it be required to lay off upon the line AD, Fig. 42, a length equal to AB; the two curves being tangent at the point A. Separate the points of the dividers a distance B1 such that the chord subtended will nearly equal the arc. Starting at B, step off 1, 2, 3, etc., along the line BA until the point nearest to the point of tangency is reached. Then, without lifting the dividers from the paper, step off from this last point the same number of divisions along the line AD. Then AE will be practically equal to AB. Care should be taken that the points 1, 2, 3, etc., where the dividers rest, are exactly on the line.

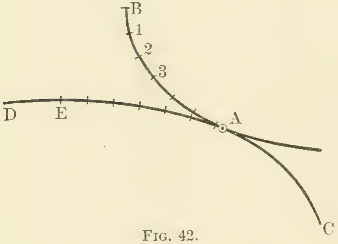


FIG. 42.

CONSTRUCTION OF CERTAIN PLANE CURVES

67. An ellipse may be generated by a point moving in the same plane, so that the sum of its distances from two fixed points shall be constantly equal to a given straight line. The two fixed points are the *foci*.

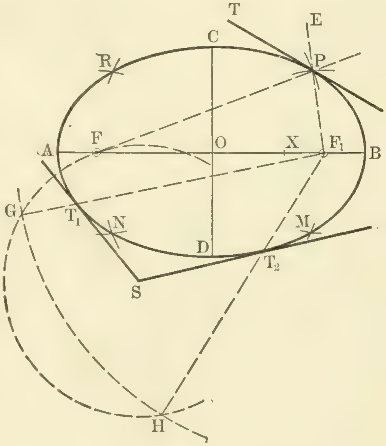


FIG. 43.

To construct an ellipse. Let F and F_1, Fig. 43, be the two foci, and AB the given line, so placed that $AF = BF_1$.

Take any point as X between F and F_1. With F as a center and AX as a radius, describe an arc. With F_1 as a center and BX as a radius, describe a second arc, intersecting the first in the points M and P. These will be points of the required curve, since

$$PF + PF_1 = AX + BX = AB.$$

In the same way all the points may be constructed. A and B are evidently points of the curve, since

$$AF + AF_1 = BF_1 + AF_1 = AB;$$

also

$$BF_1 + BF = AB.$$

The point O, midway between the foci, is the *center* of the curve. The line AB, passing through the foci and terminating in the curve, is the *transverse* or *major axis* of the curve.

The points A and B are the *vertices* of the curve. CD, perpendicular to AB at its middle point, is the *conjugate* or *minor axis* of the curve.

If the two axes are given, the foci may be constructed thus: With C, the extremity of the conjugate axis, as a center, and OA, the semi-transverse axis as a radius, describe an arc cutting AB in F and F_1. These points will be the foci, for

$$CF + CF_1 = 2\,OA = AB.$$

To construct a tangent to an ellipse at a point on the curve. Let P be the given point. Draw the focal radii PF and PF_1. The line PT, bisecting the angle FPE, is the required tangent.

To construct a tangent to an ellipse from a point without the curve. Let S be the given point. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points G and H. The lines F_1G and F_1H intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

68. A parabola may be generated by a point moving in the same plane, so that its distance from a given point shall be constantly equal to its distance from a given straight line.

The given point is the *focus*, the given straight line the *directrix*.

If through the focus a straight line be drawn perpendicular to the directrix, it is the *axis* of the parabola; and the point in which the axis intersects the curve is the *vertex*.

To construct a parabola. Points on the curve may be constructed from the definition thus: Let F, Fig. 44, be the focus, and EK the directrix. Through F draw FD perpendicular to EK. It will be the *axis*. The point V, midway between F and D, is a point of the curve, and is the *vertex*. Take any point on the axis, as X, and erect the perpendicular XP to the axis. With F as a center, and DX as a radius, describe an arc cutting XP in the two points P and M. These will be points of the curve, since

$$FP = DX = OP, \text{ also } FM = DX.$$

In the same way all the points may be constructed.

To construct a tangent to a parabola at a point on the curve. Let P be the given point. Draw FP, and the line PO perpendicular to EK. The line PT, bisecting the angle FPO, is the required tangent.

To construct a tangent to a parabola from a point without the curve. Let S be the given point. With S as center and SF as radius, strike an arc cutting the directrix in the points G and H. Lines through these points parallel to the axis DY of the parabola, cut the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

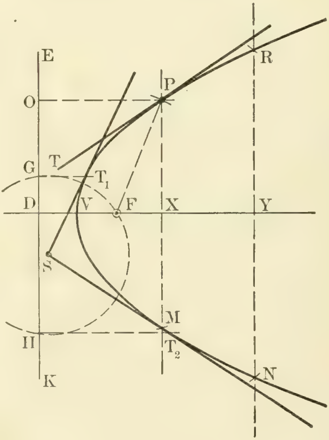


Fig. 44.

69. A hyperbola may be generated by moving a point in the same plane, so that the difference of its distances from two fixed points shall be equal to a given line.

The two fixed points are the *foci.*

To construct a hyperbola. Let F and F_1, Fig. 45, be the two foci, and AB the given line. Select any point, as X, on the line AB produced, and with F as a center and AX as a radius, strike an arc. Then with F_1 as center and BX as radius, strike an arc cutting the first arc in the points P and M. These will be points on the curve, since $AX - BX = AB$; and hence $FP - F_1P = AB$, and $FM - F_1M = AB$. If F_1 is taken as center and AX as radius, and then F as center with BX as radius, we shall get points R and N on the other branch of the hyperbola. By selecting other points than X on the line AB produced, and proceeding in a similar manner, any number of points on the curve may be determined.

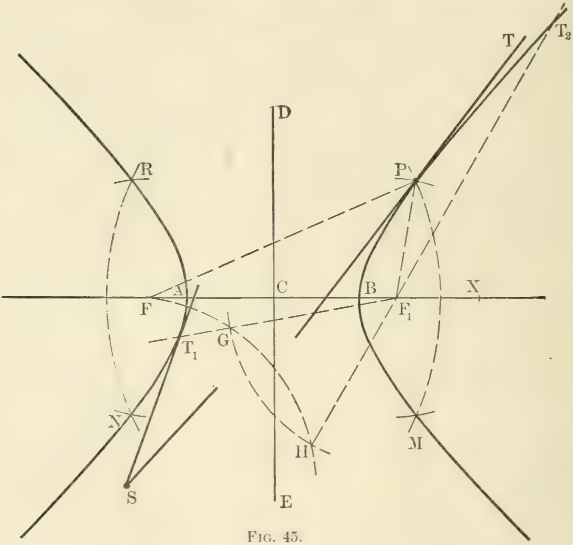


Fig. 45.

The points A and B are the *vertices* of the hyperbola. The point C, midway between the foci, is the *center*, and AB is the *transverse axis*. A perpendicular, DE, to the transverse axis at the center is the *indefinite conjugate axis*. It evidently does not intersect the curve.

To construct a tangent to the hyperbola at a point on the curve. Let P be the given point. Draw the focal radii FP and F_1P. The line PT, bisecting the angle FPF_1, is the required tangent.

To construct a tangent to the hyperbola from a point without the curve. Let S be the given point. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points G and H. Then F_1G and F_1H intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the tangents.

If tangents are passed to the hyperbola through the center C, the points of contact will be at infinity, and the lines are called *asymptotes*.

THE HELIX

70. If a point be moved uniformly around a straight line, remaining always at the same distance from it, and having at the same time a uniform motion in the direction of the line, it will generate a curve of double curvature, called a *helix*.

The straight line is the *axis* of the curve.

71. To construct the projections of the helix. Since all the points of the curve are equally distant from the axis, the projection of the curve on a plane perpendicular to this axis will be the circumference of a circle.

Thus, let *m*, Fig. 46, be the horizontal, and *m'n'* the vertical projection of the axis, and P the generating point, and suppose that while the point moves once around the axis, it moves through the vertical distance *m'n'* (called the *pitch* of the helix); *prqs* will be the horizontal projection of the curve.

To determine the vertical projection, divide *prqs* into any number of equal parts, as sixteen, and also the line *m'n'* into the same number, as in the figure. Through these points of division draw lines parallel to the ground line. Since the motion of the point is uniform, while it moves one eighth of the way round the axis it will ascend one eighth of the distance $m'n'$, and be horizontally projected at x, and vertically at x'. When the point is horizontally projected at r, it will be vertically projected at r'; and in the same way the points y', q', etc., may be determined, and $p'r'q's'$ will be the required vertical projection.

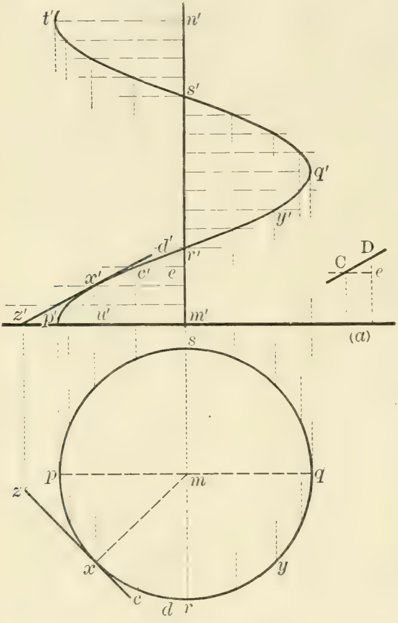


FIG. 46.

72. To determine the slope of the helix. It is evident from the nature of the motion of the generating point, that in generating any two equal portions of the curve, it ascends the same vertical distance; that is, any two elementary arcs of the curve will make equal angles with the horizontal plane. Thus, if CD (a), Fig. 46, be any element of the curve, the angle which it makes with the horizontal plane will be D*C*e, or the angle at the base of a right-angled triangle of which C$e = cd$, Fig. 46, is the base, and De the altitude. But from the nature of the motion, Ce is to De as any arc px is to the corresponding ascent $u'x'$. Hence, if we rectify the arc *xp* (Art. 63), and with this as a base construct a right-angled triangle, having $x'u'$ for its altitude, the angle at the base will be the angle which the arc, or its tangent at any point, makes with the horizontal plane. Therefore,

73. To construct the projections of a tangent to the helix at a given point, as X, we draw *xz*, Figs. 46 and 46 *a*, tangent to the circle *pxr* at *x*; it will be the horizontal projection of the required tangent. On this, from *x*, lay off the rectified arc *xp* to *z* (Art. 64); *z* will be the point where the tangent pierces H, and $z'x'$ will be its vertical projection.

74. To find the length of any given portion of the helix. Since the angle which a tangent to the helix makes with the horizontal plane is constant, and since each element of the curve is equal to the hypotenuse of a right-angled triangle of which the base is its horizontal projection, the angle at the base is the constant angle, and the altitude is the ascent of the point while generating the element, it follows that when the helix is rolled out on its tangent, the sum of the elements, or length of any portion of the curve, will be equal to the hypotenuse of a right-angled triangle, of which the base is its horizontal projection rectified, and the altitude is the ascent of the generating point while generating the portion considered. Thus the length of the arc *p*X, Figs. 46 and 46 *a*, is equal to the length of the portion of the tangent ZX.

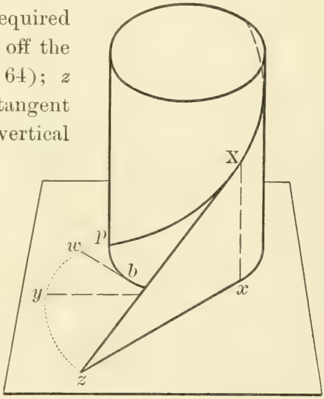


FIG. 46 *a*. Development of Helix.

$$Xbp = Xz = \sqrt{\overline{Xx}^2 + \overline{xz}^2} = \sqrt{\overline{Xx}^2 + \overline{xbp}^2}$$

CLASSIFICATION OF LINES AND SURFACES

- **Lines**
  - Straight lines
  - Plane curves (circle, ellipse, parabola, hyperbola, spirals, roulettes, etc.)
  - Space curves (helix, spherical epicycloid, loxodrome, etc.)
- **Surfaces**
  - **Ruled Surfaces** (Generated by Straight Lines)
    - **Planes**
    - **Single-curved Surfaces** (Developable)
      - All positions of generatrix parallel (cylinders)
      - All positions of generatrix intersect in common point (cones)
      - Consecutive positions of generatrix intersect two and two; no three positions intersecting in common point; may be generated by a system of tangents to line of double curvature (convolutes)
    - **Warped Surfaces** No two consecutive positions of generatrix in same plane; hence, non-developable.
      - Plane directer and two linear directrices
        - Directrices both straight (hyperbolic paraboloid)
        - One straight and one curved directrix (conoid)
        - Directrices both curved (cylindroid)
      - Three linear directrices
        - Directrices all straight (hyperboloid of one nappe)
        - One curve, two straight line directrices
        - Two curves, one straight line directrix (corne de vache)
        - Directrices all curved
      - Surfaces generated by a straight line touching a space curve and another straight line, and making a constant angle with a fixed plane (helicoid)
  - **Double-curved Surfaces** (Generated by Curved Lines)
    - **Surfaces of Revolution** (sphere, ellipsoid of revolution, torus, paraboloid of revolution, etc.)
    - **Surfaces** of double curvature not generated by a revolving line nor by a straight line (ellipsoid; elliptical paraboloid; elliptical hyperboloid)

GENERATION AND CLASSIFICATION OF SURFACES

75. Generation of surfaces. A surface *may be generated by the continued motion of a line.* The moving line is the *generatrix* of the surface; and the different positions of the generatrix are the *elements.*

If the generatrix be taken in any position, and then be moved to its next position on the surface, these two positions are *consecutive positions* of the generatrix, or *consecutive elements* of the surface, and may practically be regarded as one element.

76. Two great classes of surfaces. The form of the generatrix, and the law which directs its motion, determine the nature and class of the surface.

Surfaces may be divided into *two* general classes.

First. *Those which can be generated by straight lines, or which have rectilinear elements.* These are **ruled surfaces.**

Second. *Those which can be generated only by curves, and which can have no rectilinear elements.* These are **double-curved surfaces.**

77. There are **three classes of ruled surfaces**:

(1) **Planes**: which may be generated by *a straight line moving so as to touch another straight line, and having all its positions parallel to its first position.*

(2) **Single-curved surfaces**, *or developable surfaces*: which may be generated by *a straight line moving so that any two of its consecutive positions shall be in the same plane.*

(3) **Warped surfaces**: which may be generated by *a straight line moving so that no two of its consecutive positions shall be in the same plane.*

78. There are **three classes of single-curved surfaces:**

(1) **Cylinders**: those in which *all the positions* of the rectilinear generatrix are parallel.

(2) **Cones**: those in which *all the positions* of the rectilinear generatrix intersect in a common point.

(3) **Convolutes**: those in which the consecutive positions

of the rectilinear generatrix intersect *two and two*, no three positions intersecting in a common point.

CYLINDRICAL SURFACES, OR CYLINDERS

79. Cylinders in general. Single-curved surfaces of the first kind are *cylindrical surfaces*, or *cylinders*. Every cylinder may be generated by moving *a straight line so as to touch a curve, and have all its positions parallel.*

The moving line is the *rectilinear generatrix*. The curve is the *directrix*. The different positions of the generatrix are the *rectilinear elements of the surface.*

Thus, in Fig. 47, if the straight line MN be moved along the curve *mlo*, having all its positions parallel to its first position, it will generate a cylinder.

If the cylinder be intersected by any plane not parallel to the rectilinear elements, the curve of intersection may be taken as a *directrix*, and any rectilinear element as the generatrix, and the surface be regenerated. This curve of intersection may also be the *base of the cylinder.*

The intersection of the cylinder by one of the planes of projection, as the horizontal, is usually taken as the base. If this base have a center, the straight line through it, parallel to the rectilinear elements, is the *axis of the cylinder.*

A definite portion of the surface included by two parallel planes is sometimes considered; in which case the lower curve of intersection is the *lower base*, and the other the *upper base.*

Cylinders are distinguished by the name of their bases; as a cylinder with a circular base, a cylinder with an elliptical base, etc.

If the rectilinear elements are perpendicular to the plane of the base, the cylinder is a *right cylinder*, and the base a *right section.*

A cylinder may also be generated by moving the curvilinear directrix, as a generatrix, along any one of the rectilinear elements, as a directrix, the curve remaining always parallel to its first position.

If the curvilinear directrix be changed to a straight line, the cylinder becomes a plane.

It is manifest that if a plane parallel to the rectilinear elements intersects the cylinder, the lines of intersection will be rectilinear elements which will intersect the base.

80. Projecting cylinders. It will be seen that the projecting lines of the different points of a curve (Art. 56) form a right cylinder, the base of which, in the plane of projection, is the projection of the curve.

These cylinders are respectively the *horizontal* and *vertical projecting cylinders* of the curve, and by their intersection determine the curve.

81. To construct the projections of a cylinder. A cylinder is represented by projecting one or more of the curves of its surface and its principal rectilinear elements.

When these elements are not parallel to the horizontal plane, it is usually represented thus: Draw the base as *mlo*, Fig. 47, in the horizontal plane. Tangent to this draw straight lines *lx* and *kd* parallel to the horizontal projection of the generatrix; these will be the horizontal projections of the extreme rectilinear elements as seen from the point of sight, thus forming the horizontal projection of the cylinder. Draw tangents to the base perpendicular to the ground line as *mm′*, *oo′*; through the points *m′* and *o′* draw lines *m′n′* and *o′s′* parallel to the vertical projection of the generatrix, thus forming the vertical projection of the cylinder, *m′o′* being the vertical projection of the base.

82. To assume a rectilinear element, we have simply to draw a line parallel to the rectilinear generatrix through any point of the base, as U, Fig. 47. Then *ub* and *u′b′* will be the projections of the element.

To assume a point of the surface. Let *p*, Fig. 47, be the horizontal projection of the point. Through *p* draw *pq* parallel to *lx*; it will be the horizontal projection of two elements, one of which pierces H at *q* and the other at *u*; and *q′y′* and *u′b′* will be the vertical projections of these elements, one on the upper and one on the lower side of the cylinder; and *p′* and *r′* are the vertical projections of the two points of the surface horizontally projected at *pr*.

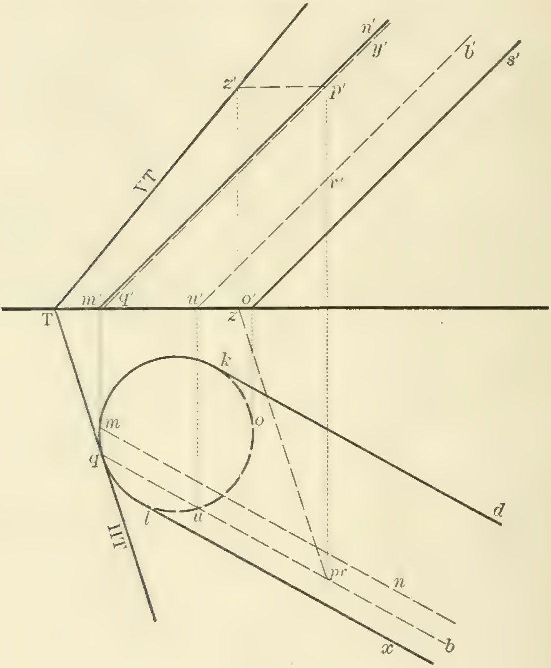


FIG. 47.

CONICAL SURFACES, OR CONES

83. Single-curved surfaces of the second kind are *conical surfaces*, or *cones.*

Every cone may be generated by moving a straight line so as continually to touch a given curve and pass through a given point not in the plane of the curve.

The moving line is the *rectilinear generatrix;* the curve, the *directrix;* the given point, the *vertex* of the cone; and the different positions of the generatrix, the *rectilinear elements.*

The generatrix being indefinite in length will generate two parts of the surface on different sides of the vertex which are called *nappes;* one the *upper*, the other the *lower*, *nappe.*

Thus, if the straight line MS, Fig. 48, move along the curve *mlo* and continually pass through S, it will generate a cone.

If the cone be intersected by any plane not passing through the vertex, the curve of intersection may be taken as a *directrix* and any rectilinear element as a *generatrix*, and the cone be regenerated. This curve of intersection may also be the *base* of the cone. The intersection of the cone by the horizontal plane is usually taken as the base.

If a definite portion of either nappe of the cone included between two parallel planes is considered, it is called a *frustum of the cone;* one of the limiting curves being the *lower* and the other the *upper base* of the frustum.

Cones are distinguished by the names of their bases; as a cone with a circular base, a cone with a parabolic base, etc.

If the rectilinear elements all make the same angle with a straight line passing through the vertex, the cone is a *right cone*, the straight line being its *axis.*

If the curvilinear directrix of a cone be changed to a straight line, or if the vertex be taken in the plane of the curve, the cone will become a plane.

If the vertex be removed to an infinite distance, the cone will evidently become a cylinder.

If a cone be intersected by a plane through the vertex, the lines of intersection will be rectilinear elements intersecting the base.

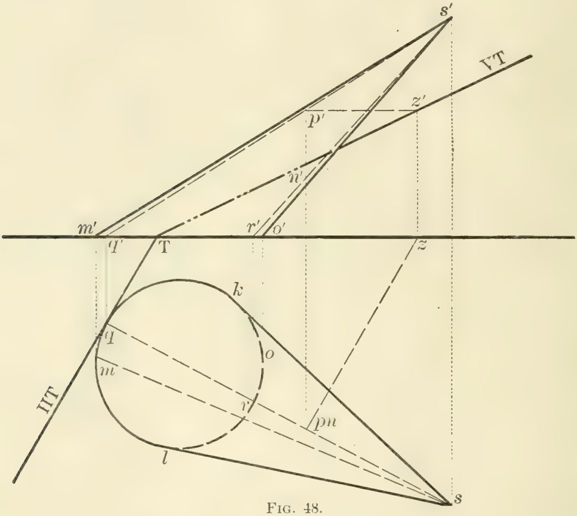


Fig. 48.

84. To construct the projections of a cone. A cone is represented by projecting the vertex, one of the curves on its surface, and its principal rectilinear elements. Thus let S, Fig. 48, be the vertex. Draw the base *mlo* in the horizontal plane, and tangents to this base through *s*, as *sl* and *sk*, thus forming the horizontal projection of the cone. Draw tangents to the base perpendicular to the ground line, as *mm'*, *oo'*, and through *m'* and *o'* draw the straight lines *m's'* and *o's'*, thus forming the vertical projection of the cone.

85. To assume a rectilinear element, we have simply to draw through any point Q of the base a straight line to the vertex. Thus qs and $q's'$ represent an element of the cone through Q.

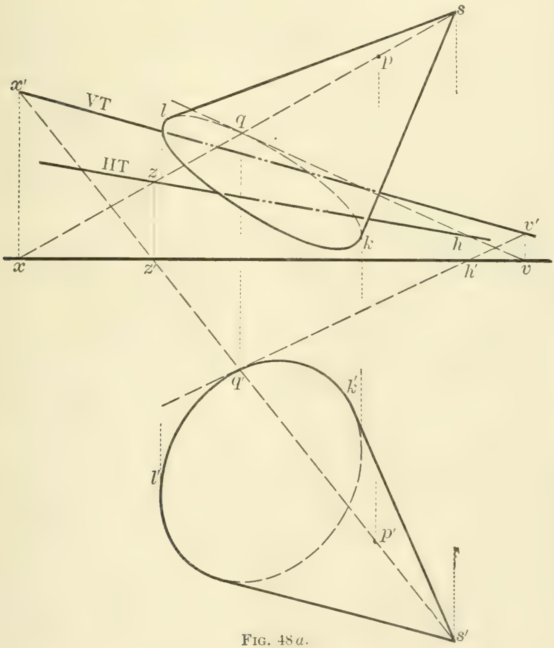


FIG. 48 *a*.

To assume a point on the surface, we first assume one of its projections, as the horizontal, pass an element through it, and find the vertical projection of the point on the vertical projection of the element, as in Art. 82.

Note. While a cylinder or cone is usually represented with the base in one of the planes of projection, it is sometimes more convenient to assume the base in other positions. Thus in Fig. 48 *a*, the base may be any plane curve or space curve. An element SQ or point P of the surface would be represented as shown.

Planes Tangent to Surfaces

86. A plane is tangent to a surface when it has at least one point in common with the surface, through which, if any intersecting plane be passed, the straight line cut from the plane will be tangent to the line cut from the surface at the point. This point is the *point of contact.*

It follows from this definition that the tangent plane is *the locus of, or place in which are to be found, all straight lines tangent to lines of the surface at the point of contact;* and since any two of these straight lines are sufficient to determine the tangent plane, we have the following general rule for passing a plane tangent to any surface at a given point : **Draw any two lines of the surface intersecting at the point. Tangent to each of these, at the same point, draw a straight line. The plane of these two tangents will be the required plane.**

87. A straight line is normal to a surface at a point, when *it is perpendicular to the tangent plane at that point.*

A plane is *normal to a surface* when *it is perpendicular to the tangent plane at the point of contact.* Since any plane passed through a normal line will be perpendicular to the tangent plane, there may be an infinite number of normal planes to a surface at a point, while there can be but one normal line.

To pass a line normal to a surface at a given point, we first determine the traces of the tangent plane at that point, and then pass a line through the point perpendicular to the plane.

88. Proposition XXXI. **The plane tangent at any point of a surface with rectilinear elements must contain the rectilinear**

elements that pass through the point of contact. For the tangent to each rectilinear element is the element itself (Art. 60), and this tangent must lie in the tangent plane (Art. 86).

89. Proposition XXXII. **A plane which contains a rectilinear element of a single-curved surface, and its consecutive one, will be tangent to the surface at every point of this element, or all along the element.** For if through *any point* of the element *any intersecting plane* be passed, it will intersect the consecutive element in a point consecutive with the first point. The straight line joining these two points will lie in the given plane and be tangent to the line cut from the surface (Art. 60). Hence the given plane will be tangent at the assumed point.

This element is the *element of contact.*

90. Proposition XXXIII. *Conversely,* **If a plane be tangent to a single-curved surface, it must, in general, contain two consecutive rectilinear elements.** For if through any point of the element contained in the tangent plane (Art. 88) we pass an intersecting plane, it will cut from the surface a line, and from the plane a straight line, which will have two consecutive points in common (Art. 86). Through the point consecutive with the assumed point draw the element consecutive to the first element; it must lie in the plane of the second point and first element (Art. 75); that is, in the tangent plane.

91. It follows from these principles, that if a plane be tangent to a single-curved surface, and the element of contact be intersected by any other plane, the straight line cut from the tangent plane will be tangent to the line cut from the surface. Hence, if the base of the surface be in the horizontal plane, *the horizontal trace of the tangent plane must be tangent to this base,* at the point in which the element of contact pierces the horizontal plane; and the same principle is true if the base lie in the vertical plane.

PROBLEMS RELATING TO PLANES TANGENT TO CYLINDERS AND CONES

92. The solution of the following problems depends mainly upon the principles that a plane tangent to a single-curved surface is tangent all along a rectilinear element (Art. 89); and that if such surface and tangent plane be intersected by any plane, the lines of intersection will be tangent to each other (Art. 86).

93. PROBLEM 22. **To pass a plane tangent to a cylinder at a given point on the surface.**

Let the cylinder be given as in Art. 81, Fig. 47, and let P be the point, assumed as in Art. 82.

Analysis. Since the required plane must contain the rectilinear element through the given point, and its horizontal trace must be tangent to the base at the point where this element pierces the horizontal plane (Art. 91), we draw the element, and at the point where it intersects the base, a tangent; this will be the horizontal trace. The vertical trace must contain the point where this element pierces the vertical plane, and also the point where the horizontal trace intersects the ground line. A straight line joining these two points will be the vertical trace. When this element does not pierce the vertical plane within the limits of the drawing, we draw through any one of its points a line parallel to the horizontal trace; it will be a line of the required plane, and pierce the vertical plane in a point of the vertical trace.

Construction. Draw the element PQ; it pierces H at q. At this point draw qT tangent to mlo; it is the required horizontal trace. Through P draw PZ parallel to qT (Art. 27); it pierces V at z', and z'T is the vertical trace.

$mm'n'$ is the plane tangent to the surface along the element MN.

94. PROBLEM 23. **To pass a plane through a given point without the surface tangent to a cylinder.**

Let the cylinder be given as in the preceding problem.

Analysis. Since the plane must contain a rectilinear element, if we draw a line through the given point, parallel to the rectilinear elements of the cylinder, it must lie in the tangent plane, and the point in which it pierces the horizontal plane will be one point of the horizontal trace. If through this point we draw a tangent to the base, it will be the required horizontal trace. A line through the point of contact, parallel to the rectilinear elements, will be the element of contact, and the vertical trace may be constructed as in the preceding problem. Or a point of this trace may be obtained by finding the point in which the auxiliary line pierces the vertical plane.

Two or more tangent planes may be passed if two or more tangents can be drawn to the base from the point in which the auxiliary line pierces the horizontal plane.

Let the construction be made in accordance with the above analysis.

95. PROBLEM 24. **To pass a plane tangent to a cylinder and parallel to a given straight line.**

Let the cylinder be given as in Fig. 49, and let RS be the given line.

Analysis. Since the required plane must be parallel to the rectilinear elements of the cylinder as well as to the given line, if a plane be passed through this line parallel to a rectilinear element, it will be parallel to the required plane, and its traces parallel to the required traces (Prop. XII, Art. 14). Hence a tangent to the base, parallel to the horizontal trace of this auxiliary plane, will be the required horizontal trace. The element of contact and vertical trace may be found as in the preceding problem.

Construction. Through RS pass the plane F, parallel to MN. Tangent to *mlo* and parallel to HF, draw HT; it is the horizontal trace of the required plane, and VT, parallel to VF, is

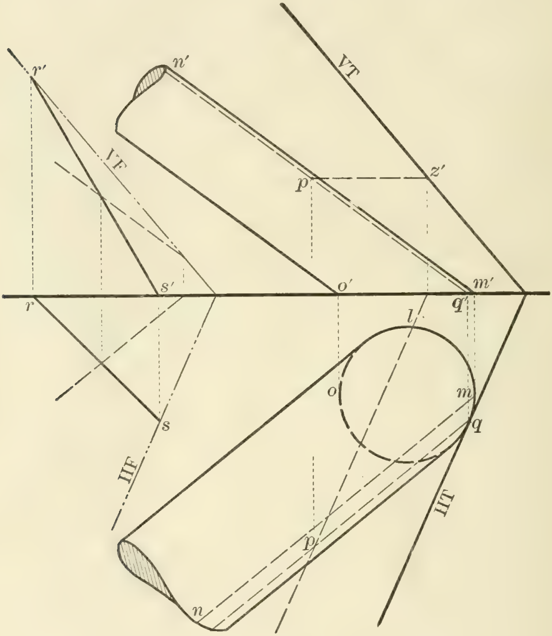


FIG. 49.

the vertical trace. QP is the element of contact. The point z', determined as in the preceding problem, will aid in verifying the accuracy of the drawing.

When more than one tangent can be drawn parallel to HF, there will be more than one solution to the problem.

96. Problem 25. **To pass a plane tangent to a right cylinder with a circular base having its axis parallel to the ground line, at a given point on the surface.**

Let GK, Fig. 50, be the axis of the cylinder, and p' the vertical projection of the point. To determine its horizontal projection, at p' erect a perpendicular to V. Through this pass the profile plane dTc' (Art. 19). It intersects the cylinder in the circumference of a circle, of which M is the center. This circumference will intersect the perpendicular in two points of the surface. Revolve this plane about Tc' until it coincides with V (Arts. 19–22). M falls at m''. With m'' as a center, draw the revolved position of the circle cut from the cylinder; p_1'' and p_2'' will be the revolved positions of the two points, and p_1 and p_2 their horizontal projections. Let the plane be passed tangent to the cylinder at P_2.

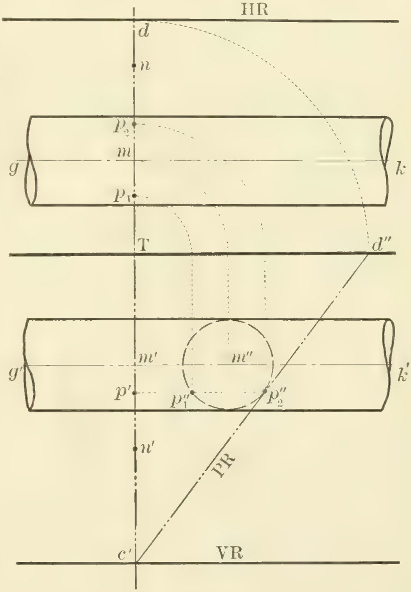


Fig. 50.

Analysis. Since the plane must contain a rectilinear element, it will be parallel to the ground line, and its traces, therefore, parallel to the ground line (Prop. VIII, Art. 14). If through the given point a plane be passed perpendicular to the axis, and a tangent be drawn to its intersection with the cylinder at the point, it will be a line of the required plane. If through the points in which this line pierces the planes of projection lines be drawn parallel to the ground line, they will be the required traces.

Construction. Through P pass the plane T, and revolve it as above. At p_2'' draw $p_2''c'$ tangent to the circle; it is the revolved position of the tangent. When the plane is revolved to its primitive position, this tangent pierces V at c' and H at d, Td being equal to Td''; and HR and VR are the required traces.

97. PROBLEM 26. **To pass a plane tangent to a right cylinder with a circular base having its axis parallel to the ground line, through a given point without the surface.**

Let the cylinder be given as in Fig. 50, and let N be the given point.

Analysis. Since the required plane must contain a rectilinear element, it must be parallel to the ground line. If through the given point a plane be passed perpendicular to the axis, it will cut from the cylinder a circumference equal to the base; and if through the point a tangent be drawn to this circumference, it will be a line of the required plane, and the traces may be determined as in the preceding problem.

Since two tangents can be drawn, there may be two tangent planes.

Let the construction be made in accordance with the analysis.

98. PROBLEM 27. **To pass a plane parallel to a given straight line, and tangent to a right cylinder with a circular base having its axis parallel to the ground line.**

Let the cylinder be given as in Fig. 51, and let MN be the given line.

Analysis. Since the plane must be parallel to the axis, if through the given line we pass a plane parallel to the axis, it will be parallel to the required plane and to the ground line.

A plane perpendicular to the ground line will cut from the cylinder a circumference; from the required tangent plane a tangent to this circumference (Art. 86), and from the parallel plane a straight line parallel to the tangent. If, then, we construct the circumference, and draw to it a tangent parallel to the intersection of the parallel plane, this tangent will be a line of the required plane, from which the traces may be found as in the preceding problems.

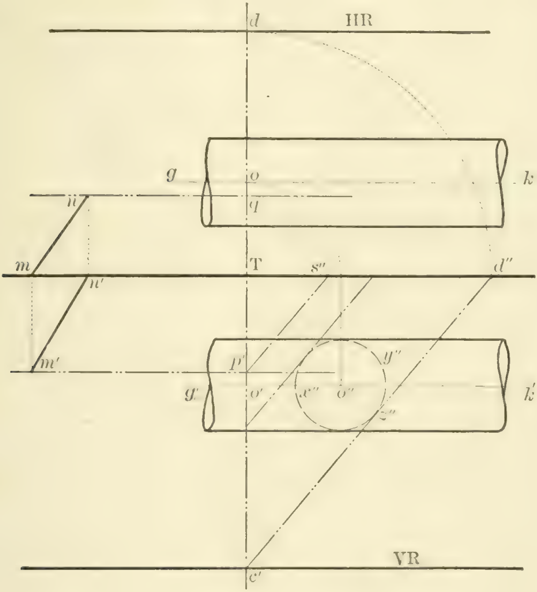


FIG. 51.

Construction. Through m' and n draw the lines $m'p'$ and nq, parallel to the ground line. They will be the traces of the parallel plane (Art. 33). Let $d\mathrm{T}c'$ be the plane perpendicular to the ground line. It cuts from the cylinder a circle whose center is O, and from the parallel plane a straight line which pierces H at q and V at p' (Art. 40). Revolve this plane about Tc' until it coincides with V; $x''y''z''$ will be the revolved position of the circle, and $s''p'$ that of the line cut from the parallel plane; $c'd''$ tangent to $x''y''z''$, and parallel to $s''p'$, will be the revolved position of a line of the required plane, which, in its true position, pierces H at d, and V at c', and HR and VR are the traces of the required plane.

Since another parallel tangent can be drawn, there will be two solutions.

99. Problem 28. **To pass a plane tangent to a cone, through a given point on the surface.**

Let the cone be given as in Fig. 48, and let P, assumed as in Art. 85, be the given point.

Analysis. The required plane must contain the rectilinear element, passing through the given point (Art. 88). If, then, we draw this element (Art. 85), and at the point where it pierces the horizontal plane draw a tangent to the base, it will be the horizontal trace of the required plane, and points of the vertical trace may be found as in the similar case for the cylinder.

Let the construction be made in accordance with the analysis.

If the base of the cone lies in an oblique plane, as in Fig. 48 a, the tangent QV will, of course, not be the horizontal trace. In this case the tangent plane is determined by QV and the element QS.

100. Problem 29. **Through a point without the surface of a cone to pass a plane tangent to the cone.**

Let the cone be given as in Fig. 52, and let P be the given point.

Analysis. Since the required plane must contain a rectilinear element, it will pass through the vertex; hence, if we join the given point with the vertex by a straight line, it will be a line of the required plane, and pierce the horizontal plane in a point of the required horizontal trace, which may then be drawn tangent to the base. If the point of contact with the base be joined to the vertex by a straight line, it will be the element of contact. The vertical trace may be found as in the preceding problems.

If more than one tangent can be drawn to the base, there will be more than one solution.

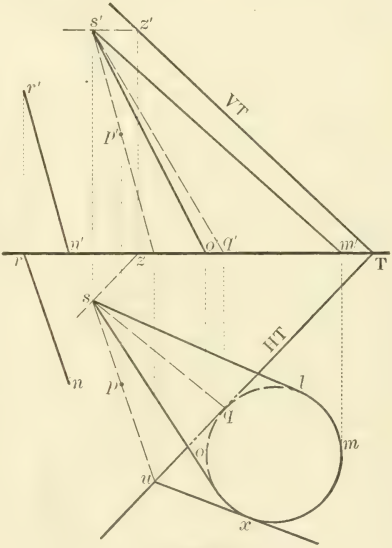


FIG. 52.

Construction. Join P with S by the line PS; it pierces H at *u*. Draw *uq* tangent to *mlo*; it is the required horizontal trace. SQ is the element of contact which pierces V in a point of the vertical trace. SZ, parallel to HT, pierces V at *z*′, and *z*′T is the vertical trace.

ux will be the horizontal trace of a second tangent plane through P.

101. PROBLEM 30. **To pass a plane tangent to a cone and parallel to a given straight line.**

Let the cone be given as in Fig. 52, and let NR be the given straight line.

Analysis. If through the vertex we draw a line parallel to the given line, it must lie in the required plane, and pierce the horizontal plane in a point of the horizontal trace. Through this point draw a tangent to the base; it will be the required horizontal trace, and the element of contact and the vertical trace may be found as in the preceding problem.

When more than one tangent can be drawn to the base, there will be more than one solution.

If the parallel line through the vertex pierces the horizontal plane within the base, no tangent can be drawn, and the problem is impossible.

Let the construction be made in accordance with the analysis.

Ex. 97. Given a right circular cone with its axis parallel to the ground line; pass a plane tangent to the cone at a point on the surface.

Ex. 98. Pass a plane tangent to the above cone through a point outside the surface.

Ex. 99. Pass a plane tangent to the cone of Ex. 97 and parallel to a given straight line.

Ex. 100. Given a plane T, and a point P lying in that plane. Construct the projection of a line PX, passing through P, making an angle of 30° with H and lying in the plane T.

Ex. 101. Given an oblique plane and an oblique line, to pass another plane through the line making an angle of 45° with the given plane. Analyze and construct.

POINTS IN WHICH SURFACES ARE PIERCED BY LINES

102. The points in which a straight line pierces a surface are easily found by passing through the line any plane intersecting the surface. It will cut from the surface a line which will intersect the given line in the required points. This auxiliary plane should be so chosen as to intersect the surface in the simplest line possible.

If the given surface be a cylinder, a plane through the straight line parallel to the rectilinear elements should be used. It will intersect the cylinder in one or more rectilinear elements, which will intersect the given line in the required points.

If the given surface be a cone, the auxiliary plane should pass through the vertex.

If the given surface be a sphere, the auxiliary plane should pass through the center, and the line of intersection will be a great circle of the sphere.

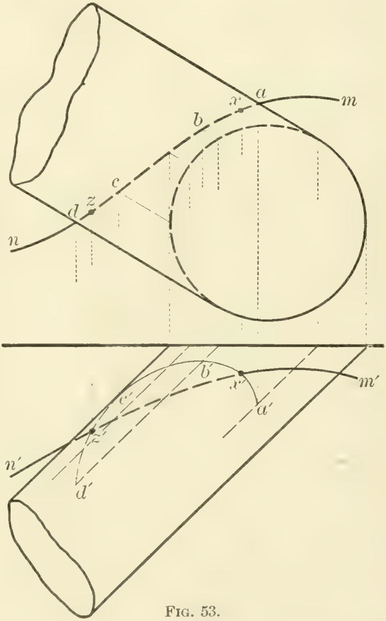


FIG. 53.

103. The points in which a *curved* line pierces a surface may be found by using the horizontal or vertical projecting cylinder (Art. 80) of the line. This will cut from the given surface a curved line which will contain the required piercing point. Thus, let it be required to find the points in which the line MN, Fig. 53, pierces the surface of the cylinder. The horizontal projecting cylinder of MN cuts the given cylinder in a curve horizontally projected in *abc*, and vertically projected in $a'b'c'$. The line MN cuts this curve in the points X and Z, which are the required piercing points.

Ex. 102. Find the point in which a curved line pierces an oblique plane.

Ex. 103. Find the points in which a curved line pierces a cone.

INTERSECTION OF CYLINDERS AND CONES. DEVELOPMENTS

104. The solution of the problem of the intersection of surfaces consists in finding two lines, one on each surface, which intersect. The point of intersection will lie in both surfaces, and will therefore be a point of their line of intersection. A series of such points will determine the curve.

105. To find the intersection of a plane with a surface, we intersect the plane and surface by a system of auxiliary planes. Each plane will cut from the given plane a straight line, and from the surface a line, the *intersection of which will be points of the required line.* The system of auxiliary planes should be so chosen as to cut from the surface the simplest lines; a rectilinear element if possible, or the circumference of a circle, etc.

To draw a tangent to the curve of intersection at a given point.

This tangent must lie in the intersecting plane, that is, in the plane of the curve (Art. 60). It must also lie in the plane tangent to the surface at the given point (Art. 86). Hence, *if we pass a plane tangent to the given surface at the given point and determine its intersection with the intersecting plane, this will be the required tangent.*

106. PROBLEM 31. **To find the intersection of a right cylinder with a circular base by a plane.**

Let *mlo*, Fig. 54, be the base of the cylinder, and *c* the horizontal, and $c'd'$ the vertical projection of the axis; then *mlo* will be the horizontal, and $s'a'n'u'$ the vertical projection of the cylinder (Art. 81). Let T be the intersecting plane.

Analysis. Intersect the cylinder and plane by a system of auxiliary planes parallel to the axis and also to the horizontal trace of the given plane. These will cut from the cylinder rectilinear elements, and from the plane straight lines parallel to the horizontal trace. The intersection of these lines will be points of the required curve (Art. 105).

This curve, as also the intersection of any cylinder or cone with a circular or an elliptical base, by a plane cutting all the rectilinear elements, is an ellipse.

Construction. Draw xy parallel to HT; it will be the horizontal trace of one of the auxiliary planes; yy' is its vertical trace. It intersects the cylinder in two elements, one of which pierces H at x and the other at z, and $f'x'$ and $b'z'$ are their vertical projections. It intersects the plane in the straight line XY (Art. 40), and X and Z, the intersections of this line with the two elements, are points of the curve. In the same way any number of points may be found.

If a point of the curve on any particular element be required, we have simply to pass an auxiliary plane through this element. Thus, to construct the point on $(o, n'u')$, draw the trace ow, and construct as above the point O.

Since this point lies on the curve, and also on the extreme element, and since no point of the curve can be vertically projected outside of $n'u'$, the vertical projection of the curve must be tangent to $n'u'$ at o'. Or, the reason may be given thus, in many like cases: If a tangent be drawn to the curve at O, it will lie in the plane tangent to the surface at O (Art. 86); and since this tangent plane is perpendicular to the vertical plane, the tangent will be vertically projected into its trace $n'u'$, which must therefore be tangent to $m'o'l'$ at o' (Art. 61). Also $a's'$ must be tangent to $m'o'l'$ at m'.

If a plane be passed through the axis perpendicular to the intersecting plane, it will evidently cut from the plane a straight line which will bisect all the chords of the curve perpendicular to it, and this line will be the transverse axis of the ellipse; *lc* is the horizontal trace of such a plane, and *gg′* the vertical. It cuts the given plane in KG (Art. 40), and the cylinder in two elements horizontally projected at *k* and *l*, and KL is the transverse axis, and K and L are the vertices of the ellipse. K is the lowest, and L the highest, point of the curve.

Since the curve lies on the surface of the cylinder, its horizontal projection will be in the base *mlo*.

To draw a tangent to the curve at any point, as X; draw *xr* tangent to *xol* at *x*; it is the horizontal trace of a plane tangent to the cylinder at X (Art. 93). This plane intersects the plane T in a straight line, which pierces H at *r*; and since X is also a point of the intersection, RX will be the required tangent (Art. 105). If a tangent be drawn at each of the points determined as above, the projections of the curve can be drawn, with great accuracy, tangent to the projections of these tangents, at the projections of the points of tangency.

The part of the curve MXO, between the points M and O, lies in front of the extreme elements (m, $a's'$) and (o, $n'u'$) and is seen, and therefore $m'x'o'$ is drawn full, and $m'l'o'$ broken.

To represent the curve in its true dimensions, let its plane be revolved about HT until it coincides with H. The revolved position of each point, as X, at x_1, will be determined as in Art. 34; k_1 and l_1 will be the revolved position of the vertices, rx_1 will be the revolved position of the tangent, and $l_1x_1k_1$ the ellipse in its true size.

107. Development. Since the plane tangent to a single-curved surface in general contains two consecutive rectilinear elements (Art. 90), it will contain the elementary portion of the surface generated by the generatrix in moving from the first element to the second. Now if the surface be rolled over until the consecutive element following the second comes into the tangent plane, the portion of the plane limited by the first and third elements will equal the portion of the surface limited by the same elements. If we continue to roll the surface on the tangent plane until any following element comes into it, the portion of the plane included between this element and the first will be equal to the portion of the surface limited by the same elements. Therefore, if a single-curved surface be rolled

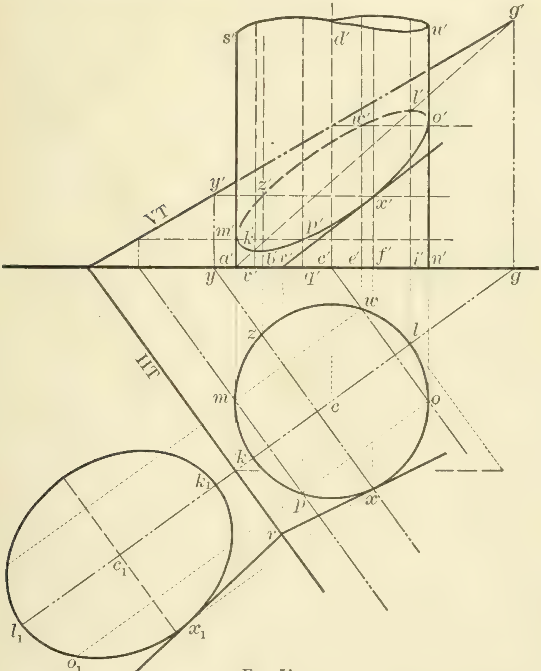


FIG. 54.

over on any one of its tangent planes until each of its rectilinear elements has come into this plane, *the portion of the plane thus touched by the surface, and limited by the extreme elements*, will be a plane surface equal to the given surface, and is *the development of the surface.*

As a plane tangent to a warped surface cannot contain two consecutive rectilinear elements (Art. 140), the elementary surface limited by these two elements cannot be brought into a plane without breaking the continuity of the surface. **A warped surface, therefore, cannot be developed.**

Neither can a double-curved surface be developed, as any elementary portion of the surface will be limited by two curves, and cannot be brought into a plane without breaking the continuity of the surface.

108. In order to determine the position of the different rectilinear elements of a single-curved surface, as they come into the tangent plane, or *plane of development*, it will always be necessary to find some curve upon the surface which will develop into a straight line, or circle, or some simple known curve, upon which the rectified distances between these elements can be laid off.

109. PROBLEM 32. **To develop a right cylinder with a circular base, and trace upon the development the curve of intersection of the cylinder by an oblique plane.**

Let the cylinder and curve be given as in the preceding problem, and let the plane of development be the tangent plane at K.

Analysis. Since the plane of the base is perpendicular to the element of contact of the tangent plane, it is evident that as the cylinder is rolled out on this plane, this base will develop into a straight line on which we can lay off the rectified distances between the several elements and then draw them each parallel to the element of contact.

Points of the development of the curve are found by laying off on the development of each element, from the point where it meets the rectified base, a distance equal to the distance of the point from the base.

Construction. The plane of development being coincident with the plane of the paper, let *k*K, Fig. 54 *a*, be the element of contact. Draw *kk* perpendicular to *k*K, and lay off *kk* equal to the rectified circumference *kxwk*, Fig. 54. It will be the development of the base. Lay off *kp* equal to the arc *kp*, and draw *p*P parallel to *k*K. It is the development of the element which pierces H at *p*. Likewise for each of the elements lay off *px*, *xo*, etc., equal respectively to the rectified arcs *px*, *xo*, etc., in Fig. 54, and draw *x*X, *o*O, etc. The portion of the plane included between *k*K and *k*K will be the development of the cylinder.

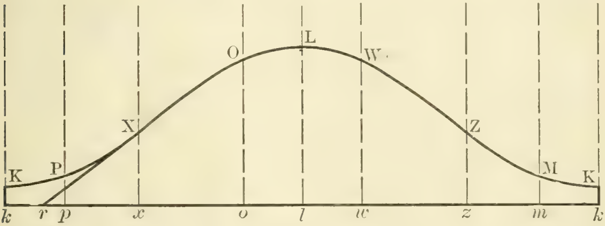


FIG. 54 *a*.

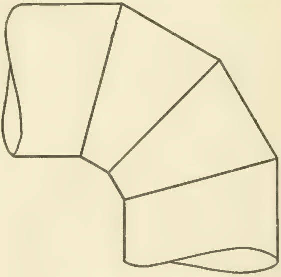


FIG. 55.—Stovepipe Elbow.

On *k*K lay off *k*K equal to *v'k'*, and K will be one point of the developed curve; on *p*P lay off *p*P equal to *q'p'*, and P will be a second point; and thus each point may be determined, and K - - L - - K will be the developed curve. The line rx, the *sub-tangent*, will take the position rx on the developed base, and Xr will be the tangent at X in the plane of development. This must be tangent at X, since the tangent after development must contain the same two consecutive points which it contains in space, and therefore be tangent to the development of the curve (Art. 60).

110. Problem 33. **To find the intersection of an oblique cylinder by a plane.**

Let the cylinder be given as in Fig. 56, and let T be the intersecting plane.

Analysis. Intersect the cylinder and plane by a system of auxiliary planes parallel to the rectilinear elements and perpendicular to the horizontal plane. These planes will each intersect the cylinder in two rectilinear elements, and the plane in a straight line, the intersection of which will be points of the curve.

Construction. Let eq, parallel to li, be the horizontal trace of an auxiliary plane: qq' will be its vertical trace. It intersects the cylinder in two elements, one of which pierces H at r, the other at s, and these are vertically projected in $r'g'$ and $s'h'$, and horizontally in eq. It intersects the plane T in a straight line, which pierces H at e, and is vertically projected in $e'q'$, and horizontally in eq. These lines intersect in Z and Y (Prop. XXVIII, Art. 16), points of the curve. In the same way any number of points may be determined.

The auxiliary planes, being parallel, must intersect T in parallel lines, the vertical projections of which will be parallel to $e'q'$.

By the plane whose horizontal trace is mn, the points U and X are determined. The vertical projection of the curve must be tangent to $m'n'$ at u'. The points in which the horizontal projection is tangent to li and kf are determined by using these lines as the traces of auxiliary planes.

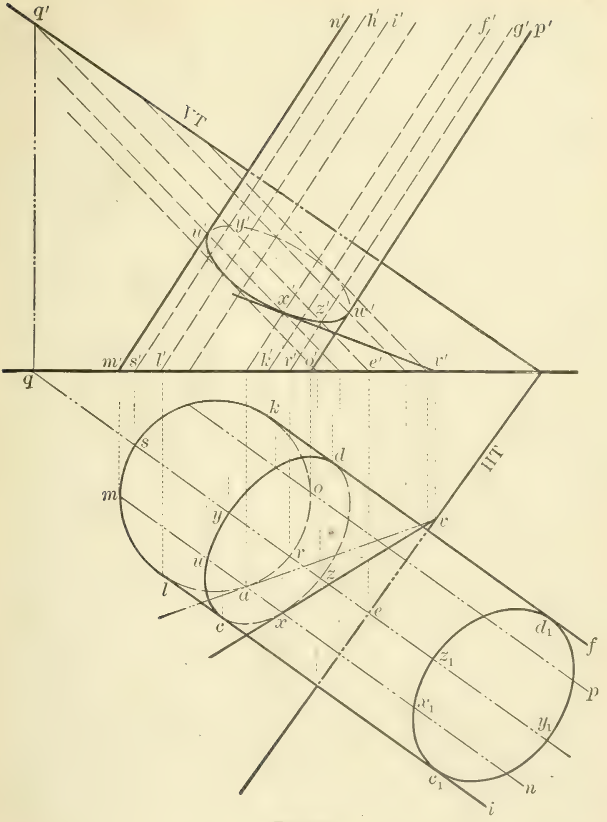


Fig. 56.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cylinder at X; *av* is its horizontal trace, and *vx* the horizontal, and *v'x'* the vertical projection of the tangent.

A sufficient number of points and tangents being thus determined, the projections of the curve can be drawn with accuracy. The part *cyd* is full, being the horizontal projection of that part of the curve which lies above the extreme elements LI and KF. For a similar reason, $u'x'w'$ is full.

To show the curve in its true dimensions, revolve the plane about HT until it coincides with H. The revolved position of each point may be found as in Art. 34, and $c_1y_1d_1z_1$ will be the curve in its true size.

The section in this case is a *right section*, since the cutting plane was assumed perpendicular to the elements.

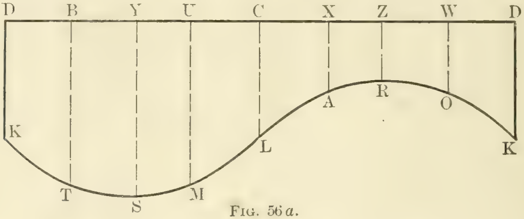


FIG. 56 *a*.

111. If it be required to develop the cylinder on a tangent plane along any element, as KDF, we first make a right section as above. We know this will develop into a straight line perpendicular to KDF. On this we lay off the rectified arcs of the section included between the several elements, and then draw these elements parallel to KD.

The developed base, *or any curve on the surface*, may be traced on the plane of development, Fig. 56 *a*, by laying off on each element, from the developed position of the point where it intersects the right section, the distance from this point to the point where the element intersects the base or curve. A line through the extremities of these distances will be the required development.

Ex. 104. Pass a plane perpendicular to the elements of an oblique cylinder; find the curve of right section, and then develop that portion of the cylindrical surface lying between the plane of right section and the base.

112. PROBLEM 34. **To find the intersection of a right cone with a circular base, by a plane.**

Let the cone be given as in Fig. 57, and let T be the given plane, the vertical plane being assumed perpendicular to it.

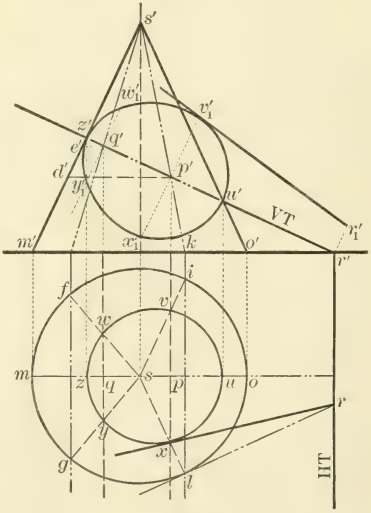


FIG. 57.

Analysis. Intersect the cone by a system of planes through the vertex and perpendicular to the vertical plane. The element cut from the cone by each plane (Art. 105) will intersect the straight line cut from the given plane in points of the required curve.

Construction. Let lk be the horizontal trace of an auxiliary plane; ks' will be its vertical trace. It intersects the cone in two elements, one of which pierces H in l, and the other in i, horizontally projected in ls and is respectively. It intersects the given plane in a straight line perpendicular to V, vertically projected at p', and horizontally in xv; hence x and v are the horizontal projections of two points of the curve, both vertically projected at p'.

In the same way any number of points can be found, as yq', wq', etc.

The plane whose horizontal trace is mo, perpendicular to T, intersects it in a straight line vertically projected in VT, which evidently bisects all chords of the curve perpendicular to it, and is therefore an axis of the curve. This plane cuts from the cone the elements SM and SO, which are intersected by the axis in the points Z and U, which are the vertices.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cone at X; lr is its horizontal trace. It intersects T in $(rx, r'p')$, which is therefore the required tangent (Art. 105). The horizontal projection, $uxzv$, can now be drawn. $u'z'$ is its vertical projection.

To represent the curve in its true dimensions, we may revolve it about HT until it coincides with H, or about VT until it coincides with V, and determine it as in Art. 34. Otherwise, thus: Revolve it about UZ until its plane becomes parallel to V; it will then be vertically projected in its true dimensions (Art. 58). The points U and Z being in the axis, will be projected at u' and z' respectively. X will be vertically projected at x'_1, $p'x'_1$ being equal to px; Y at y'_1, $q'y'_1$ being equal to qy; W at w'_1, etc.; and $u'v'_1z'x'_1$ will be the curve in its true size.

113. If a right cone with a circular base be intersected by a plane, as in Fig. 57, making a *less angle* with the plane of the base than the elements do, the curve of intersection is an *ellipse*. If it make the *same angle*, or is parallel to one of the elements, the curve is a *parabola*. If it make a *greater angle*, the curve is a *hyperbola*. Hence these three curves are known by the general name *conic sections*.

114. PROBLEM 35. **To develop a right cone with a circular base.**

Let the cone and its intersection by an oblique plane be given as in the preceding problem, Fig. 57, and let the plane of development be the plane tangent along the element MS; the half of the cone in front being rolled to the left, and the other half to the right.

Analysis. Since the base of the cone is everywhere equally distant from the vertex, as the cone is rolled out, each point of this base will be in the circumference of a circle described with the vertex as a center, and a radius equal to the distance from the vertex to any point of the base. By laying off on this circumference the rectified arc of the base, contained between any two elements, and drawing straight lines from the extremities to the vertex, we have, in the plane of development, the position of these elements. Laying off on the proper elements the distances from the vertex to the different points of the curve of intersection, and tracing a curve through the extremities, we have the development of the curve of intersection.

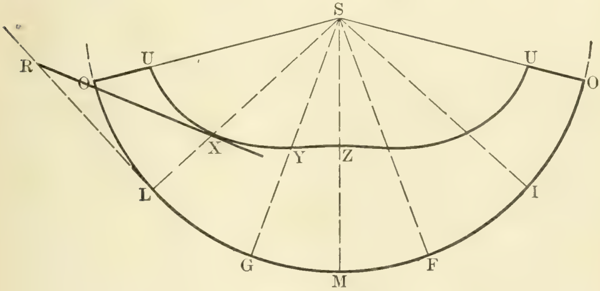


FIG. 57 *a*.

Construction. With SM equal to $s'm'$, Figs. 57 and 57 *a*, describe the arc OMO. It is the development of the base. Lay off MG equal to *mg*, and draw SG; it is the developed position of the element SG. In the same way lay off GL equal to *gl*, LO equal to *lo*, and draw SL and SO. OSM is the development of one half the cone. In like manner the other half may be developed.

On SM lay off SZ equal to $s'z'$. Z will be the position of the point in the plane of development. To obtain the true distance from S to Y, revolve SY about the axis of the cone until it becomes parallel to V, as in Art. 26; $s'e'$ will be its true length. On SG lay off SY equal to $s'e'$; on SL, SX equal to $s'd'$; on SO, SU equal to $s'u'$. Z, Y, X, and U will be the positions of these points on the plane of development, and UX - - U will be the development of the curve of intersection.

Through L draw LR perpendicular to LS, and make it equal to lr. RX will be the developed tangent.

Note. The work of rectifying the circumference of the base can be more accurately done by calculating the angle OSO from the relation,

$$\text{Angle OSO} = \frac{360\ R}{\sqrt{h^2 + R^2}}\ \textit{degrees},$$

where R is the radius of the base of the cone, and h is the altitude. Furthermore, if in Fig. 57 the base had been divided into equal parts before passing the auxiliary planes, the arcs *mg*, *gl*, etc., would all be equal, and their lengths would be laid off in Fig. 57 *a* by simply dividing OMO into the same number of equal parts.

115. PROBLEM 36. **To find the intersection of any cone by a plane.**

Let the cone and plane T be given as in Fig. 58.

Analysis. Intersect the cone by a system of planes through the vertex and perpendicular to the plane of the base. Each of these planes will intersect the cone in one or more rectilinear elements, and the given plane in a straight line, the intersection of which will be points of the curve. Since these auxiliary planes are perpendicular to the plane of the base they will intersect in a straight line through the vertex perpendicular to the plane of the base, and the point in which this line pierces the cutting plane will be a point common to all the straight lines cut from this plane.

Construction. Find the point in which the perpendicular through S to the plane of the base (H in this problem) pierces T, as in Art. 28. v' is its vertical projection. The vertical projections of the lines cut from T all pass through this point.

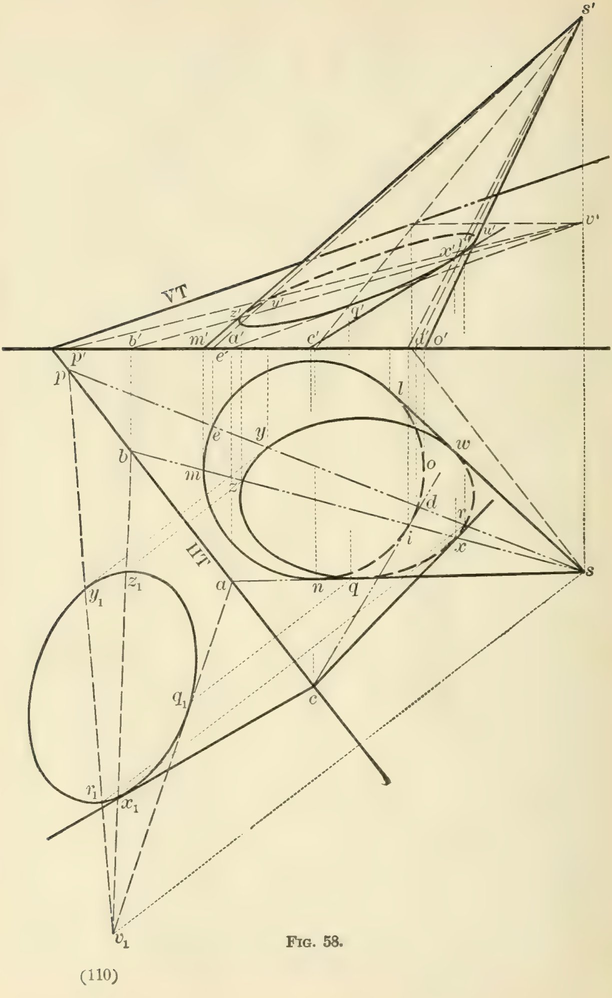
Let sp be the horizontal trace of an auxiliary plane. It intersects the cone in the elements SE and SD, and the cutting plane in the straight line (ps, $p'v'$). This line intersects the elements in R and Y, which are points of the required curve. In the same way any number of points may be found.

To find the point of the curve on any particular element, as SM, we pass an auxiliary plane through this element. sm is its horizontal trace, and Z and X are the two points on this element. The vertical projection of the curve is tangent to $s'm'$ at z', and to $s'o'$ at u'. The points q and w, in which the horizontal projection is tangent to sl and sn, are found by using as auxiliary planes the two planes whose traces are sl and sn.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cone at X. ic is its horizontal trace (Art. 99). It intersects T in CX, which is therefore the required tangent (Art. 105).

The part of the curve which lies above the two extreme elements SL and SN is seen, and therefore its projection, $wyzq$, is full. For a similar reason the projection $z'q'x'u'$, of that part of the curve which lies in front of the two extreme elements, SM and SO, is full.

To show the curve in its true dimensions, revolve the plane about HT until it coincides with H, and determine each point, as Q at q_1, as in Art. 34. Or the position of (s, v') may be found


s'
v'
u'
x'
r'
VT
z'
y'
q'
b'
m'
a'
c'
d'
o'
p'
e'
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y
w
b
o
m
z
d
r
i
x
HT
s
a
n
q
y_1
z_1
q_1
c
r_1
x_1
v_1

FIG. 58.

at v_1. Then if the points c, a, b, p, etc., be each joined with this point by straight lines, we have the revolved positions of the lines cut from the given plane by the auxiliary planes, and the points y_1, z_1, r_1, x_1, in which these lines are intersected by the perpendiculars to the axis, yy_1, zz_1, etc., are points of revolved position of the curve. x_1c is the revolved position of the tangent.

If, in Fig. 57, both traces of the cutting plane made acute angles with the ground line, how would the auxiliary planes be passed in order to find the required points most conveniently?

116. To find the intersection of any two curved surfaces, we intersect them by a system of auxiliary surfaces. Each auxiliary surface will cut from the given surfaces lines, *the intersection of which will be points of the required line* (Art. 104).

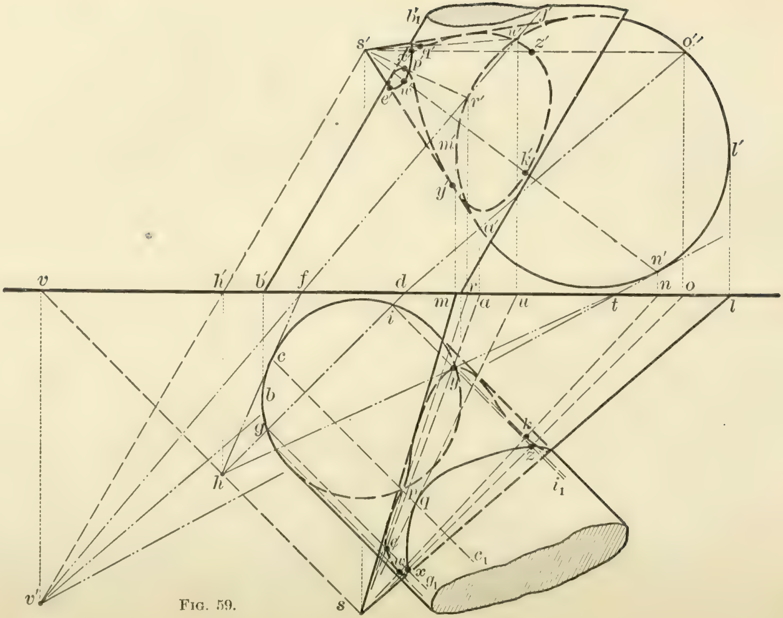
The system of auxiliary surfaces should be so chosen as to cut from the given surfaces the simplest lines, rectilinear elements if possible, or the circumferences of circles, etc.

To draw a tangent to the curve of intersection at any point, pass a plane tangent to each surface at this point. The *intersection* of these two planes will be the required *tangent*, since it must lie in each of the tangent planes (Art. 86).

In constructing this curve of intersection, great care should be taken to determine those points in which its projections are tangent to the limiting lines of the projections of the surfaces; and also those points in which the curve itself is tangent to other lines of either surface, as these points aid much in drawing the curve with accuracy.

117. PROBLEM 37. **To find the intersection of a cylinder and a cone.**

Let the surfaces be given as in Fig. 59; the base of the cylinder *bci* being in the horizontal plane, the base of the cone *m'l'o'* in the vertical plane, and S its vertex.



b′₁
s′
z′
q′
p′
w′
e′
r′
m′
y′
k′
a′
o″
z′
l′
n′
v
h′
b′
f
d
r
m
a
u
t
n
o
l
i
c
b
g
h
y
k
z
i₁
p
q
e
c₁
v
x
g₁
v′
s

FIG. 59.

Analysis. Intersect the two surfaces by a system of auxiliary planes *passing through the vertex of the cone and parallel to the rectilinear elements of the cylinder.* Each plane will intersect each of the surfaces in two rectilinear elements, the intersection of which will be points of the required curve. These planes will intersect in a straight line passing through the vertex of the cone and parallel to the rectilinear elements of the cylinder, and the point in which this line pierces the horizontal plane will be a point common to the horizontal traces of all the auxiliary planes. Also the point in which this line pierces V will be common to the vertical traces of all the auxiliary planes.

Construction. Through S draw SV parallel to the elements of the cylinder. It pierces H in h, and V in v'. Through h draw any straight line, as hd; it may be taken as the horizontal trace of an auxiliary plane, the vertical trace of which is do', passing through v', the point in which SV pierces the vertical plane. This plane intersects the cylinder in two elements which pierce H at g and i, and are horizontally projected in gg_1 and ii_1. The same plane intersects the cone in two elements which pierce V in a' and o', and are horizontally projected in as and os. These elements intersect in the points E, X, Y, and Z, which are points of the required curve. In the same way any number of points may be determined.

The vertical projection of the required curve is tangent to $a's'$ at the points e' and y'; since $a's'$ is the vertical projection of one of the extreme elements of the cone. The points of tangency on any other extreme element, either of the cone or the cylinder, may be determined by using an auxiliary plane which shall contain that element.

A tangent to the curve at any of the points thus determined may be constructed by finding the intersection of two planes, one tangent to the cylinder and the other to the cone, at this point (Art. 116).

The plane of which *hc* is the horizontal trace is tangent to the cylinder along the element CC_1, and intersects the cone in the two elements SR and SU. If at P a plane be passed tangent to the cone, it will be tangent along SP, which is also its intersection with the plane tangent to the cylinder. SP is then tangent to the curve at P, and for a similar reason SQ is tangent at Q. Hence the two projections *sp* and *sq* are tangent to *epy - - zq* at *p* and *q* respectively; and the vertical projections of the same elements will be tangent at p' and q'.

The plane of which $v't$ is the vertical trace is tangent to the cone along the element SX, and intersects the cylinder in two elements, the horizontal projections of which are tangent to *epy - - zq* at *k* and *w*.

The projections of the curve can now be drawn with great accuracy. The horizontal projection of that part which lies on the upper portion of both cylinder and cone is drawn full. Likewise the vertical projection of that part which lies on the front side of both surfaces is full.

If two of the auxiliary planes used in finding the line of intersection be passed tangent to the cylinder, and both intersect the cone, it is evident that the cylinder will penetrate the cone so as to form two distinct curves of intersection. If one intersects the cone and the other does not, a portion only of the cylinder enters the cone, and there will be a continuous curve of intersection, as in the figure.

If neither of these planes intersects the cone, and the cone lies between them, the cone will penetrate the cylinder, making two distinct curves.

If both planes are tangent to the cone, all the rectilinear elements of both surfaces will be cut.

Ex. 105. Assume a cylinder and cone with bases in H, and find the curve of intersection, showing invisible portions in each projection by dashed lines.

118. PROBLEM 38. **To find the intersection of two cylinders.** If we pass a plane through a rectilinear element of one cylinder parallel to the rectilinear elements of the other, and then intersect the cylinders by a system of planes parallel to this plane, points on the curve of intersection of the two cylinders may be found by a process similar to that of the preceding problem. The horizontal traces of all the auxiliary planes will be parallel; likewise the vertical traces.

EX. 106. Assume a cylinder with base in H, and another with base in V, and find the curve of intersection.

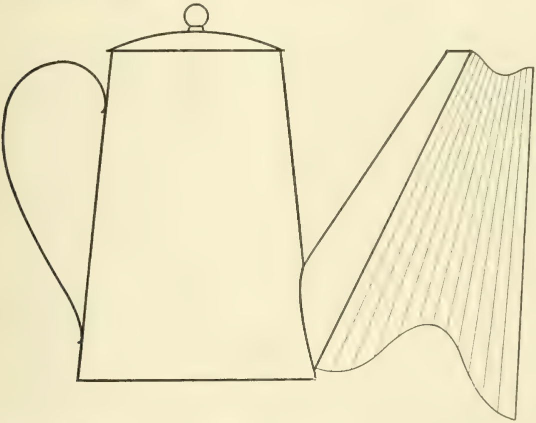


FIG. 60.—Intersection of Two Conical Surfaces; Development of Spout of Teapot.

119. PROBLEM 39. **To find the intersection of two cones,** pass a system of planes through the vertices of both cones. The straight line joining these vertices will lie in all of these planes, and pierce the horizontal plane in a point common to all the horizontal traces; and it will pierce the vertical plane in a point common to all the vertical traces.

We may ascertain whether the surfaces intersect in two distinct curves, or only one, in the same manner as in Art. 117, by passing certain auxiliary planes tangent to either surface.

Ex. 107. Assume two intersecting cones, one with base in H, the other with base in V, and find the curve of intersection.

120. Problem 40. **To find the intersection of a cone and a hemisphere.**

Let *mlo*, Fig. 61, be the base of the cone, and S its vertex at the center of the sphere, *abc* being the horizontal and $a'd'c'$ the vertical projection of the hemisphere.

Analysis. Intersect the surfaces by a system of planes passing through the vertex and perpendicular to the horizontal plane. Each plane will cut from the cone two rectilinear elements, and from the hemisphere a semicircumference, the intersection of which will be points of the required curve.

Construction. Take *sp* as the horizontal trace of one of the auxiliary planes. It intersects the cone in the two elements SP and SQ, and the hemisphere in a semicircle whose center is at S. Revolve this plane about the horizontal projecting line of S, until it becomes parallel to V. The element SP will be vertically projected in $s'p'_1$, SQ in $s'q'_1$, the semicircle in $a'd'c'$, and x'_1 and y'_1 will be the vertical projections of the revolved positions of the points of intersection. In the counter-revolution these points describe the arcs of horizontal circles and in their true position will be vertically projected at x' and y', and horizontally at x and y.

In the same way any number of points may be found.

The points of tangency u' and z' are found by using auxiliary planes which cut out the extreme elements SM and SO of the cone.

The points in which the vertical projection of the curve is tangent to the semicircle $a'd'c'$ are found by using the auxiliary plane whose trace is *st*.

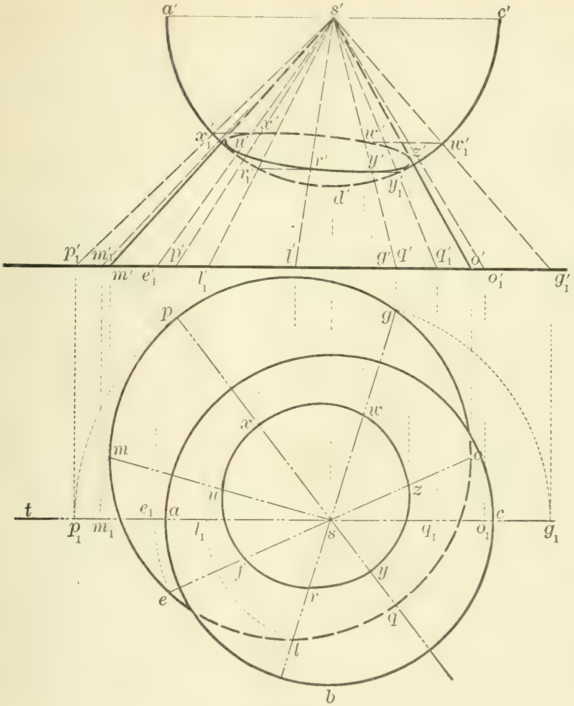


FIG. 61.

121. PROBLEM 41. **To develop an oblique cone with any base.**

Let the cone be given as in the preceding problem, Fig. 61, and let it be developed on the plane tangent along the element SP.

First method. Analysis. If the cone be intersected by a sphere having its center at the vertex, all the points of the curve of intersection will be at a distance from the vertex equal to the radius of the sphere; hence when the cone is developed this curve will develop into the arc of a circle having its center at the position of the vertex and its radius equal to that of the sphere. On this we can lay off the rectified arcs of the curve of intersection included between the several rectilinear elements (Art. 107), and then draw these elements to the position of the vertex.

The developed base, *or any curve on the surface*, may be traced on the plane of development by laying off on each element, from the vertex, the distance from the vertex to the point where the element intersects the base or curve. A line through the extremities of these distances will be the required development.

Construction. Find, as in the preceding problem, the curve XUY - - -. With S, Fig. 62, as a center and *sa* as a radius, describe the arc XUR - - -. It is the indefinite development of the intersection of the sphere and the cone. Draw SX for the position of the element SX.

To find the distance between any two points measured on the curve XUY - - -, we first develop its horizontal projecting cylinder on a plane tangent to it at X, as in Art. 109. X′U′R′X′, Fig. 62 *a*, is the development of the curve. On XUR lay off XU equal to X′U′, UR equal to U′R′, etc., and draw SU, SR, etc. These will be the positions of the elements on the plane of development. On these lay off SP equal to $s'p'_1$, SM equal to $s'm'_1$, etc., and join the points PME, etc., and we have the development of the base of the cone.

122. Second method: development by triangulation. If in Fig. 61 the elements are taken sufficiently numerous so that the chords *pm*, *me*, etc., are practically equal to the subtended arcs, the cone may be developed without the use of the auxiliary sphere, by conceiving the surface to be approximately a pyramid made up of the triangles PSM, MSE, ESL, etc.

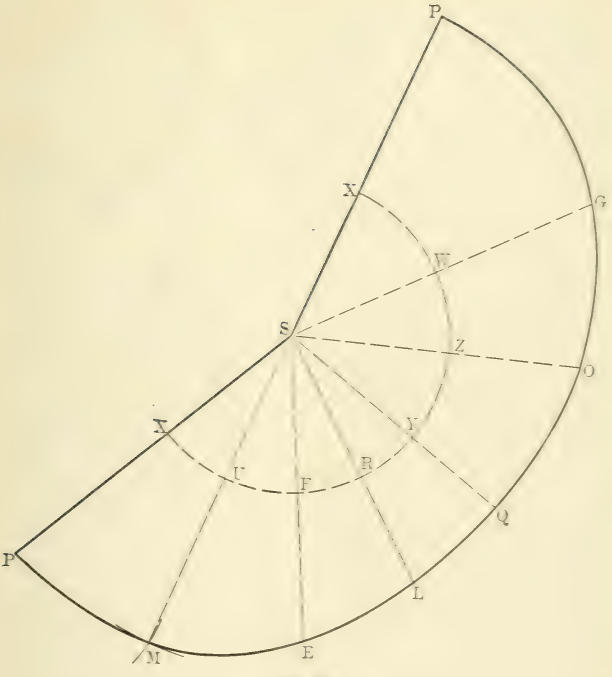


FIG. 62.

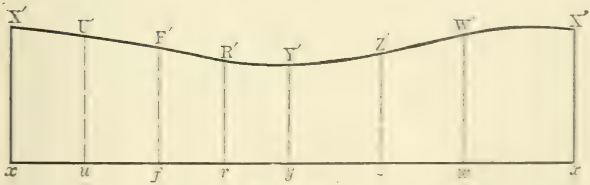


FIG. 62 a.

Let SP be laid off in its true length ($s'p'_1$). Then with S, Fig. 62, as center and a radius equal to $s'm'_1$, the true length of the element SM, strike an arc. Also with P as center and a radius equal to the chord *pm*, strike an arc intersecting the arc just drawn. This determines a triangle PSM. To the right of this triangle lay off the adjoining one SME in the same manner, continuing until all the triangles have been laid off. A smooth curve through the points PMEL, etc., will be the developed curve of the base.

Any curve of the surface may be traced on the plane of development by laying off on each element, from the vertex, the distance from the vertex to the point where the element intersects the curve, and drawing a smooth curve through the points thus determined.

Ex. 108. Analyze the problem, to find the shortest path on the surface of a cylinder or cone, between two given points on the surface.

CONVOLUTES

123. Single-curved surfaces of the third kind are called convolutes. They may be generated by drawing a system of tangents to any space curve. These tangents will evidently be rectilinear elements of a single-curved surface. For if we conceive a series of consecutive points of a space curve as *a*, *b*, *c*, *d*, etc., the tangent which contains *a* and *b* (Art. 60) is intersected by the one which contains *b* and *c* at *b*; that which contains *b* and *c*, by the one which contains *c* and *d* at *c*; and so on, each tangent intersecting the preceding consecutive one, but not the others, since no two elements of the curve not consecutive are in general in the same plane (Art. 55).

124. The helical convolute. If the curve to which the tangents are drawn is a helix, the surface may be represented thus: Let *pxy*, Fig. 64, be the horizontal, and $p'x'y'$ the vertical projection of the helical directrix. Since the rectilinear elements are all tangent to this directrix, any one may be assumed, as in Art. 73; hence XZ, YU, etc., are elements of the surface.

A point of the surface may be assumed by taking a point on any assumed element.

These elements pierce H in the points *z*, *u*, *v*, etc., and *zuvw* is the horizontal trace of the surface, and may be regarded as its base; and it is evident that if the surface be intersected by any plane parallel to this base, the curve of intersection will be equal to the base.

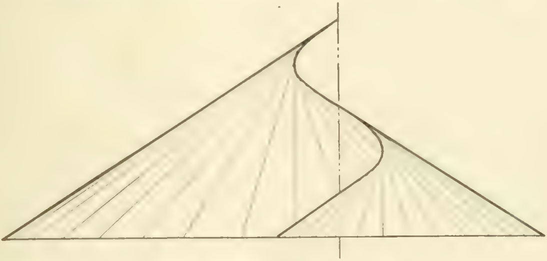


FIG. 63. — Helical Convolute.

Since each of the tangents, *xz*, *os*, etc., is equal to the rectified arc, from the springing point *p*, the curve is an *involute* of the circle *pxy*.

This surface is also known as the *Olivier Spiraloid*, and is used as the working surface in the teeth of skew bevel gears. See McCord's Kinematics of Machines, Arts. 382–385; Robinson's Principles of Mechanism, pp. 181-186; Grant's Teeth of Wheels, Arts. 175–177; American Machinist, Aug. 28, 1890.

125. PROBLEM 42. **To pass a plane tangent to a helical convolute at a given point on its surface.**

Let the surface be given as in Fig. 64, and let R be the given point (Art. 124).

Analysis. Since this is a single-curved surface, the tangent plane must be tangent all along the rectilinear element through the point of contact (Art. 89). Its horizontal trace must therefore be tangent to the base (Art. 91). If through the point where this element pierces the horizontal plane a straight line is drawn tangent to the base, it will be the required horizontal trace, and the vertical trace may be determined as in the case of the cylinder or cone.

Construction. The element RX pierces H at Z. At this point draw the tangent HT (perpendicular to *zx*); it is the horizontal trace. VT is the vertical trace.

126. Problem 43. **To pass a plane tangent to a helical convolute through a given point without the surface,** we pass a plane through the point parallel to the base and draw a tangent to the curve of intersection (Art. 124) through the point. This tangent, with the element of the surface through its point of contact, will determine the tangent plane.

127. Problem 44. **To pass a plane tangent to a helical convolute and parallel to a given straight line.**

Let the surface be given as in Fig. 64, and let MN be the given line.

Analysis. If with any point of the straight line as a vertex we construct a cone whose elements make the same angle with the horizontal plane as the elements of the surface, and pass a plane through the line tangent to this cone, it will be parallel to the required plane. The traces of the required plane may then be constructed as in Art. 95.

Construction. Take n' as the vertex of the auxiliary cone and draw $n'b$, making with the ground line an angle equal to $o's'v'$; *bca* will be the base of the cone in H. Through *m* draw *mc* tangent to *bca*. It will be the horizontal trace of the parallel plane, and HF, parallel to it and tangent to *uvw*, is the required horizontal trace. Let the student construct the vertical trace.

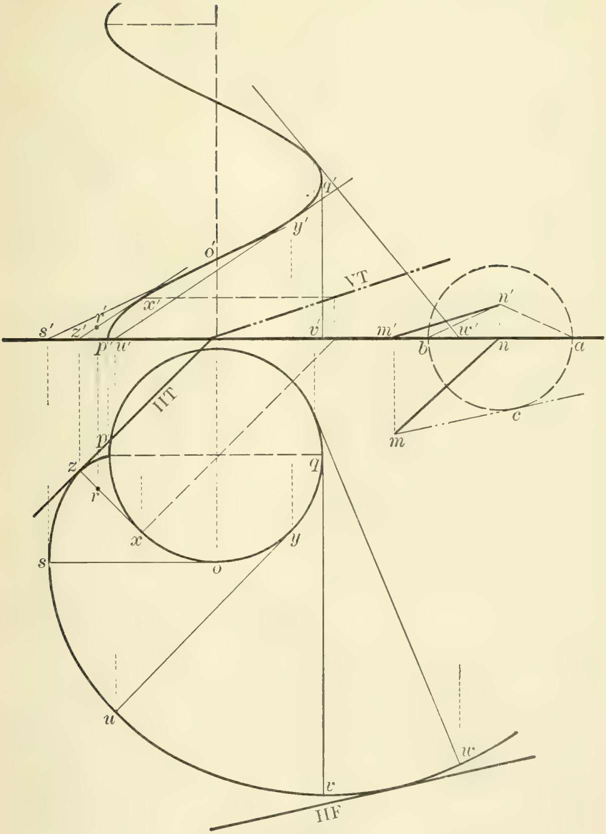


Fig. 64.

128. By an examination of the preceding problems it will be seen:

(1) That, in general, only one plane can be drawn tangent to a single-curved surface at a given point.

(2) That the number which can be drawn through a given point without the surface, and tangent along an element, will be limited.

(3) That the number which can be drawn parallel to a given straight line, and tangent along an element, is also limited.

(4) That in general a plane cannot be passed through a given straight line and tangent to a single-curved surface. If, however, the given line lies on the convex side of the surface, and is parallel to the rectilinear elements of a cylinder, or passes through the vertex of a cone, or is tangent to a line of the surface, the problem is possible.

129. PROBLEM 45. **To find the intersection of a helical convolute by a plane**, intersect the surfaces by a system of auxiliary planes tangent to the projecting cylinder of the helix. These intersect the convolute in rectilinear elements, and the plane in straight lines, the intersection of which will be points of the required curve.

130. PROBLEM 46. **To develop a helical convolute.** When the surface is laid out into a plane, the helix takes the form of a circular arc whose radius $R = r + \frac{P}{4\pi^2 r}$, in which P is the pitch of the helix, and r is the radius of the horizontal projection of the helix. The developed elements of the surface are tangent to the developed helix, and the developed base of the surface is the involute of this curve (Art. 124).

To draw the development of any curve of the surface, the proper distances may be laid off along the elements, either from the points where the elements cut the base, or from the points of tangency of the elements with the developed helix.

Ex. 109. Assume a helix whose H projection is a circle of $1\frac{1}{2}''$ diameter, and whose pitch is $4''$. (*a*) With this as a directrix construct the two projections of a helical convolute, showing eight elements in the first half convolution. (*b*) Find the intersection of this surface with an oblique plane. (*c*) Develop the surface, showing the curve of intersection upon the development. (*d*) Trace the helix upon a wooden cylinder of $1\frac{1}{2}''$ diameter, cut the development from the drawing paper, and fit the inner curve to the helix, thus causing the paper to take the true shape of the convolute.

WARPED SURFACES WITH A PLANE DIRECTER

131. There are a great variety of warped surfaces, differing from one another in their mode of generation and properties.

The most simple are those which may be generated by a straight line *generatrix*, moving so as to touch two other lines as *directrices*, and parallel to a given plane, called a *plane directer.*

Such surfaces are *warped surfaces with two linear directrices and a plane directer.*

They, as all other warped surfaces, may be represented by projecting one or more curves of the surface, and the principal rectilinear elements.

132. PROBLEM 47. **Given the directrices and plane directer of a warped surface, to construct the rectilinear element through a point of either directrix.** Pass a plane through the given point parallel to the plane directer, find the point in which the other directrix pierces this plane, and join it with the given point.

Construction. Let MN and PQ, Fig. 65, be any two linear directrices, T the plane directer, and O any point of the first directrix.

Assume any line of the plane directer, as CD (Art. 27), and through the different points of this line draw straight lines

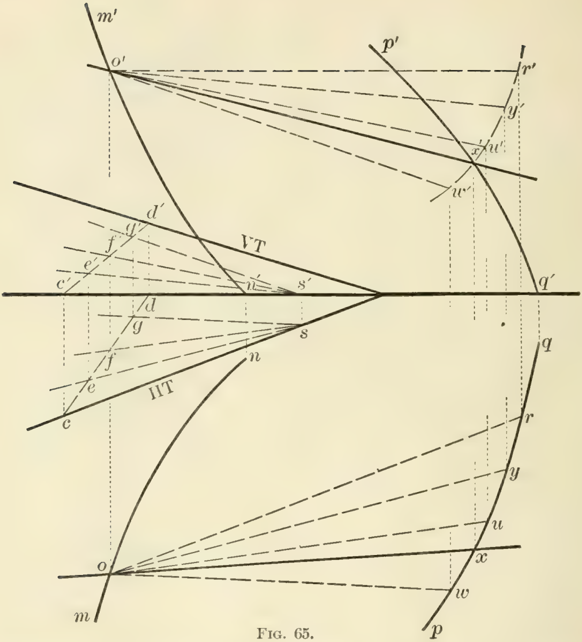


FIG. 65.

SE, SF, etc., to any point, as S, of HT. Through O draw a system of lines, OR, OY, etc., parallel respectively to SE, SF, etc. These will form a plane through O, parallel to the plane T, and pierce the horizontal projecting cylinder of PQ, in the points R, Y, W, etc. These points, being joined, will form the curve RW, which intersects PQ in X, and this is the point in which the auxiliary plane cuts the directrix PQ. OX will then be the required element. If the curve $r'w'$ should intersect $p'q'$ in more than one point, two or more elements passing through O would thus be determined.

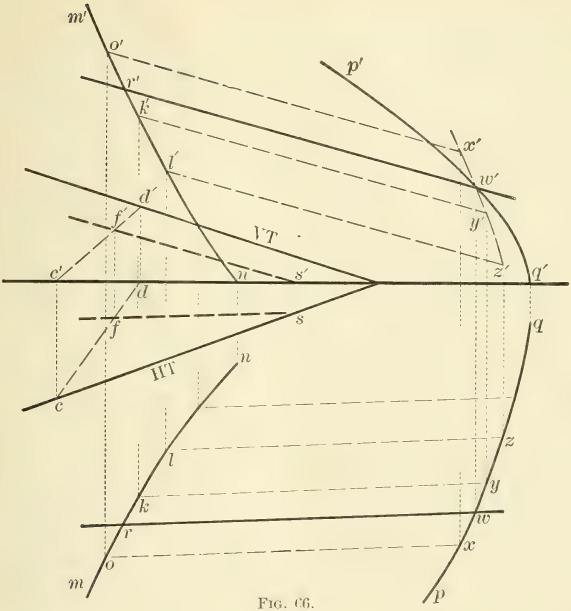


FIG. 66.

133. PROBLEM 48. **Given the directrices and plane directer of a warped surface, to construct an element parallel to a given straight line; this line being in the plane directer, or parallel to it.** Draw through the different points of either directrix, lines parallel to the given line. These form the rectilinear elements of a cylinder, parallel to the given line. If the points in which the second directrix pierces this cylinder be found, and lines be drawn through them parallel to the given line, each will touch both directrices, and be the element required.

Construction. Let the surface be given as in the preceding article and let FS, Fig. 66, be the given line. Through the points O, K, L, etc., draw OX, KY, LZ, etc., parallel to FS. These lines pierce the horizontal projecting cylinder of the directrix PQ, in the points X, Y, Z, etc., which, being joined, form the curve XY, intersecting PQ in W. Through W draw WR, parallel to FS; it is a required element, and there may be two or more as in the preceding article.

134. Classes of warped surfaces with a plane directer. If both directrices are straight lines, the surface is called a *hyperbolic paraboloid*. If one directrix is a straight line and the other a curved line, the surface is a *conoid*. If both directrices are curved, the surface is a *cylindroid*. The surfaces in Figs. 65 and 66 are cylindroids.

If the rectilinear directrix of a conoid is perpendicular to the plane directer, it is a *right conoid*, and this directrix is the *line of striction*.

The Hyperbolic Paraboloid

135. The **hyperbolic paraboloid** is so named because its intersection by a plane may be proved to be either a hyperbola or a parabola.

Its rectilinear elements may be constructed by the principles in Arts. 132 and 133. In the first case, two straight lines through the given point will determine the auxiliary plane, and the point in which it is pierced by the second directrix may be determined at once, as in Art. 43. In the second case, the cylinder becomes a plane, and the point in which it is pierced by the second directrix is also determined, as in Art. 43.

136. Proposition XXXIV. **The rectilinear elements of a hyperbolic paraboloid divide the directrices proportionally.** For these elements are in a system of planes parallel to the plane directer and to one another, and these planes divide the directrices into proportional parts at the points where they are intersected by the elements.

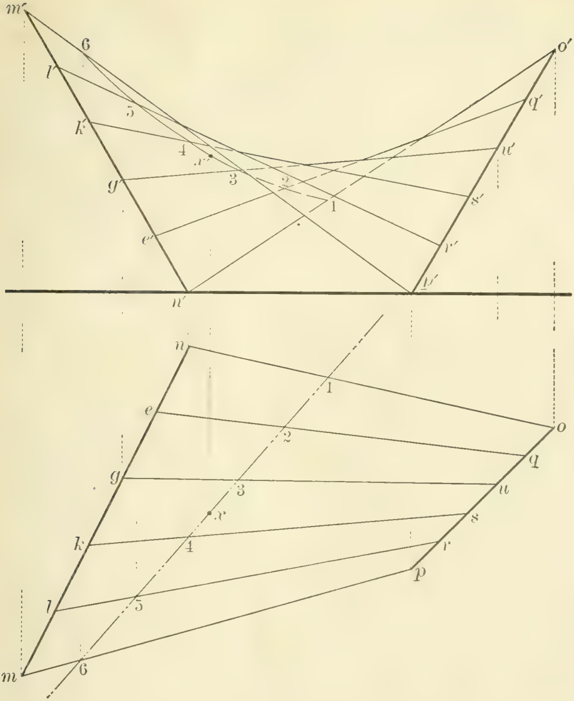


Fig. 67.

137. Proposition XXXV. *Conversely,* — **If two straight lines be divided into any number of proportional parts, the straight lines joining the corresponding points of division will lie in a system of parallel planes, and be elements of a hyperbolic paraboloid, the plane directer of which is parallel to any two of these dividing lines.**

Thus, let MN and OP, Fig. 67, be any two rectilinear directrices. Take any distance, as ml, and lay it off on mn any number of times, as ml, lk, kg, etc. Take also any distance, as pr, and lay it off on po any number of times, as pr, rs, su, qo, etc. Join the corresponding points of division by straight lines, lr, ks, gu, etc.; these will be the horizontal projections of rectilinear elements of the surface. Through m, l, k, etc., and p, r, s, etc., erect perpendiculars to the ground line, to m', l', k', etc., and p', r', s', etc., and join the corresponding points by the straight lines $l'r'$, $k's'$, $g'u'$, etc.; these will be the vertical projections of the elements.

138. PROBLEM 49. **To assume a point on any warped surface.** We first assume one of its projections, as the horizontal. If this lies on the horizontal projection of an element already drawn, its vertical projection may easily be found on the vertical projection of that element. Otherwise, pass a plane through the point perpendicular to H. It will intersect the rectilinear elements in points which, joined, will give a line of the surface, and the vertical projection of the assumed point will lie in the vertical projection of this line.

In Fig. 67, x is the assumed horizontal projection of a point on the surface of the hyperbolic paraboloid. The method of finding the location of x' will be evident from an examination of the figure.

139. PROPOSITION XXXVI. **If any two rectilinear elements of a hyperbolic paraboloid be taken as directrices, with a plane directer parallel to the first directrices, and a surface be thus generated, it will be identical with the first surface.**

To prove this we have only to prove that any point of an element of the *second generation* is also a point of the *first generation.*

Thus, let MN and OP, Fig. 68, be the directrices of the first generation, and NO and MP any two rectilinear elements. Through M draw MW parallel to NO. The plane WMP will be parallel to the plane directer of the first generation, and may be taken for it.

Let NO and MP be taken as the new directrices, and let ST be an element of the second generation, the plane directer being parallel to MN and OP. Through U, any point of ST, pass a plane parallel to WMP, cutting the directrices MN and OP in Q and R. Join QR; it will be an element of the first generation (Art. 132). Draw N*n* and Q*q* parallel to ST, piercing the plane WMP in *n* and *q*. Also draw O*o* and R*r* parallel to ST, piercing WMP in *o* and *r*; and draw *no*, intersecting MP in T, since N*n*, ST, and O*o* are in the same plane. M, *q*, and *n* will be in the same straight line, as also P, *r*, and *o*; and since MN and N*n*, intersecting at N, are parallel to the plane directer of the second generation, their plane will be parallel to it, as also the plane of PO and O*o*; hence, these planes being parallel, their intersections, M*n* and P*o*, with the plane WMP, will be parallel. Draw *qr*.

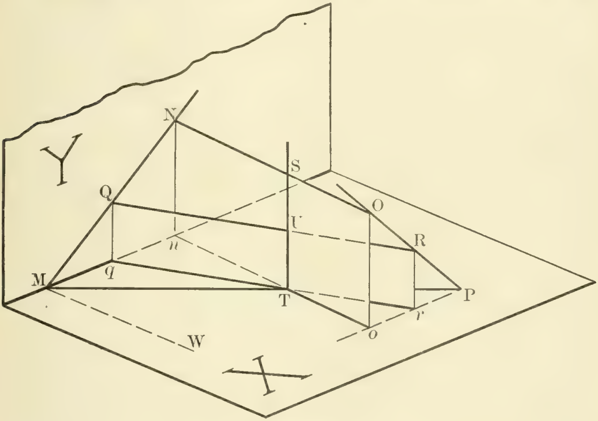


FIG. 68.

Since Qq and Nn are parallel, we have

$$MQ : QN :: Mq : qn .$$

Also $$PR : RO :: Pr : ro.$$

But (Art. 136), $$MQ : QN :: PR : RO ;$$

hence, $$Mq : qn :: Pr : ro ;$$

and the straight line, qr, must pass through T, and the plane of the three parallels, Qq, ST, and Rr, contains the element QR, which must therefore intersect ST at U. Hence, any point of a rectilinear element of the second generation is also a point of an element of the first generation, and the two surfaces are identical. It follows from this, that **through any point of a hyperbolic paraboloid two rectilinear elements can always be drawn.**

PLANES TANGENT TO WARPED SURFACES WITH PLANE DIRECTER. INTERSECTIONS

140. PROPOSITION XXXVII. **A plane tangent to a warped surface, although it contains the rectilinear element passing through the point of contact, cannot contain its consecutive element, and therefore can, in general, be tangent at no other point of the element.**

141. PROPOSITION XXXVIII. **If a plane contain a rectilinear element of a warped surface, and be not parallel to the other elements, it will be tangent to the surface at some point of this element.** For this plane will intersect each of the other rectilinear elements in a point; and these points being joined, will form a line which will intersect the given element. If at the point of intersection a tangent be drawn to this line, it will lie in the tangent plane (Art. 86). The given element, being its own tangent (Art. 60), also lies in the tangent plane. The plane of these two tangents, that is, the intersecting plane, is therefore tangent to the surface at this point (Art. 86). It is thus seen that, in general, a plane tangent to a warped surface is also an *intersecting plane*.

If the intersecting plane be parallel to the rectilinear elements, there will be no curve of intersection formed as above, and the plane will not be tangent.

142. Since a plane tangent to a warped surface must contain the rectilinear element passing through the point of contact (Art. 88), we can at once determine one line of the tangent plane. A second line may then be determined in accordance with the rule in Art. 86.

When the surface has two different generations by straight lines, the plane of the two rectilinear elements passing through the given point will be the required plane.

143. PROBLEM 50. **To pass a plane tangent to a hyperbolic paraboloid at a given point of the surface.**

Let MN and PQ, Fig. 69, be the directrices, and MP and NQ any two elements of the surface, and let O, assumed as in Art. 138, be the given point.

Analysis. Since through the given point a rectilinear element of each generation can be drawn (Art. 139), we have simply to construct these two elements and pass a plane through them (Art. 142).

Construction. Through O draw OE and OF parallel respectively to NQ and MP. These will determine a plane parallel to the plane directer of the first generation (Art. 135). This plane cuts the directrix PQ in the point U (Art. 43). Join U with O, and we have an element of the first generation (Art. 132). Let NQ and MP be taken as directrices of the second generation. Through O draw OC and OD parallel respectively to MN and PQ. They will determine a plane parallel to the plane directer of the second generation (Art. 135).

This plane cuts MP in W, and OW will be an element of the second generation, and the tangent plane is determined as in Art. 30.

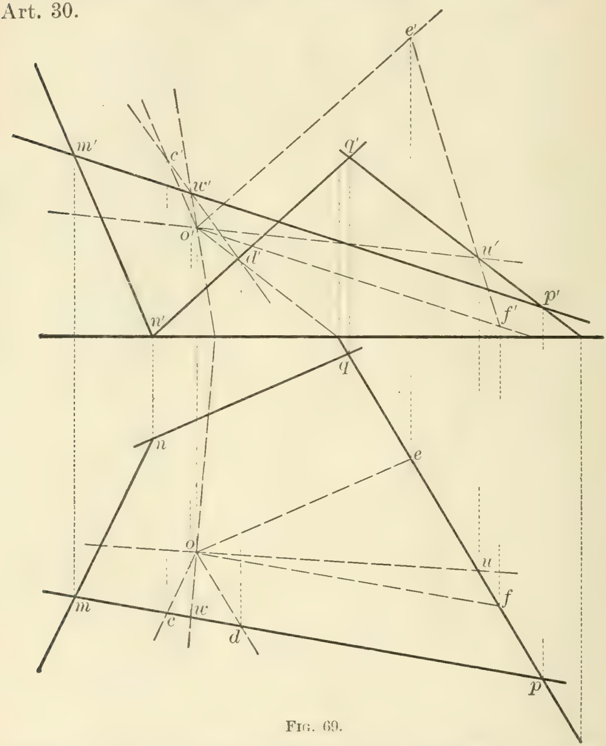


FIG. 69.

144. Two curved surfaces are tangent to each other when they have at least one point in common, through which if any intersecting plane be passed, the lines cut from the surfaces will be tangent to each other at this point. This will evidently be the case when they have a common tangent plane at this point.

145. Proposition XXXIX. **If two single-curved surfaces are tangent to each other at a point of a common element, they will be tangent all along this element.** For the common tangent plane will contain this element and be tangent to each surface at every point of the element (Art. 89).

This principle is not true with warped surfaces.

146. Proposition XL. **But if two warped surfaces, having two directrices, have a common plane directer, a common rectilinear element, and two common tangent planes, the points of contact being on the common element, they will be tangent all along this element.** For if through each of the points of contact any intersecting plane be passed, it will intersect the surfaces in two lines, which will have, besides the given point of contact, a second consecutive point in common (Art. 86). If, now, the common element be moved upon the lines cut from either surface, as directrices, and parallel to the common plane directer, into its consecutive position, containing these second consecutive points, it will evidently lie in both surfaces, and the two surfaces will thus contain two consecutive rectilinear elements. If, now, *any plane* be passed, intersecting these elements, two lines will be cut from the surfaces, having two consecutive points in common, and therefore tangent to each other; hence the surfaces will be tangent all along the common element (Art. 144).

147. Problem 51. **To pass a plane tangent to a conoid or a cylindroid at a given point of the surface.** Let MN and PQ, Fig. 70, be the two directrices of a cylindroid having V for its plane directer, and let O be the given point. At the points X and Y, in which the rectilinear element through O intersects the directrices, draw a tangent to each directrix, as XZ and YU. On these tangents as directrices move XY parallel to V. It will generate a hyperbolic paraboloid (Art. 135) having with the given surface the common element XY, and a common tangent plane at each of the points X and Y, since the plane of the two lines XY and XZ, and also that of XY and YU is tangent to both surfaces (Art. 86). The two surfaces are therefore tangent all along the common element XY (Art. 146). If, then, at O we pass a plane tangent to the hyperbolic paraboloid, it will be also tangent to the given surface at the same point. Let the student complete the construction.

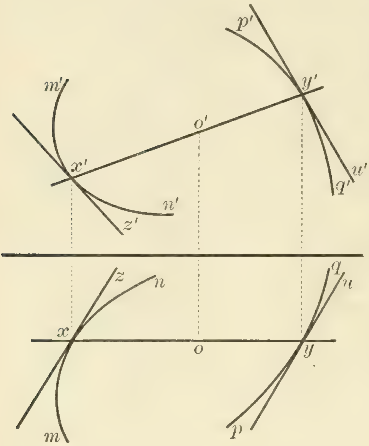


Fig. 70.

148. Problem 52. **To pass a plane through a given straight line, and tangent to a warped surface,** it is necessary only to produce the line until it pierces the surface, and through the point thus determined draw the rectilinear element of the surface. This, with the given line, will determine a plane tangent to the surface at some point of the element (Art. 141).

If there be two rectilinear elements passing through this point, each will give a tangent plane, and the number of tangent planes will depend upon the number of points in which the line pierces the surface. If the given line be parallel to a rectilinear element, it pierces the surface at an infinite distance, and the tangent plane will be determined by the two parallel lines.

149. Problem 53. **To find the intersection of any warped surface with a plane directer, by an oblique plane,** intersect by planes parallel to the plane directer. Each will cut from the surface one or more rectilinear elements, and from the plane a straight line, the intersection of which will be points of the required curve.

Or otherwise, intersect by planes perpendicular to either plane of projection; cutting elements from the surface, and straight lines from the plane.

Ex. 110. Assume two non-parallel, non-intersecting lines, AB and CD, as the directrices of a hyperbolic paraboloid, and a plane T as directer. Determine two elements of the surface by Art. 132. Then determine three other elements by Art. 137. Assume a point X of the surface not on any of these five elements (Art. 138). Construct an element MN through X (Art. 143). Construct an element OP of the second generation through X. Pass a plane S tangent to the surface at X.

Ex. 111. Find the intersection between a hyperbolic paraboloid similar to the one in Fig. 67, and an oblique plane (Art. 149).

Ex. 112. Assume a helix as the curvilinear directrix of a right conoid, and the axis of the helix as its rectilinear directrix, perpendicular to H, the horizontal plane being the plane directer. Then assume an oblique cone with base in H, and vertex in the axis of the helix. Find the intersection between the cone and conoid by passing a system of auxiliary planes in such a manner as to cut elements from both surfaces.

Ex. 113. Assume a circle in H, and a straight line parallel to the ground line as the directrices of a right conoid, with the profile plane as directer. Assume a straight line FG in such a way as to cut the surface of the conoid. Pass a plane T through FG and tangent to the surface (Art. 148). Then find the point of tangency (Art. 141). Analyze.

Ex. 114. Assume a cylindroid similar to that of Fig. 66, and any straight line in space not perpendicular to the plane directer. Pass a plane S tangent to the cylindroid and perpendicular to this line. The solution depends upon Arts. 133 and 148. Let the student analyze the problem.

Ex. 115. Assume a conoid with H as the plane directer, and pass a plane tangent to the surface at an assumed point of the surface (Art. 147). To facilitate the construction, let the projections of the curvilinear directrix be circular arcs.

The Helicoid

150. A **helicoid** is a warped surface generated by a straight line so moving that it constantly touches a helix (Art. 70), and the axis of the helix, and makes a constant angle with the axis.*

The surface may be represented by taking the axis perpendicular to the horizontal plane, Fig. 71, *o* being its horizontal, and *o′n′* its vertical projection; and constructing the helix PRQ (Art. 71), and the projections of the several elements.

151. To construct the projections of the elements of the surface. Let P be the point in H from which the helix springs, and PO, parallel to V, the first position of the generatrix; the angle *p′o′n′* being the constant angle between the elements and the axis. To construct the projections of an element through the point X on the helix, we regard the generatrix as having moved along the helix from P to X. In so doing it has risen through a vertical distance equal to $s'x'$. The point where the generatrix intersects the axis will have risen through the same distance. Hence we lay off $o'o'_1$ equal to $s'x'$; and $x'o'_1$ will be the vertical projection of the element, and *xo* its horizontal. In the same way the element (oy, o'_2y') may be assumed.

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* For a discussion of the general case of a "helicoidal surface," see McCord's Descriptive Geometry.

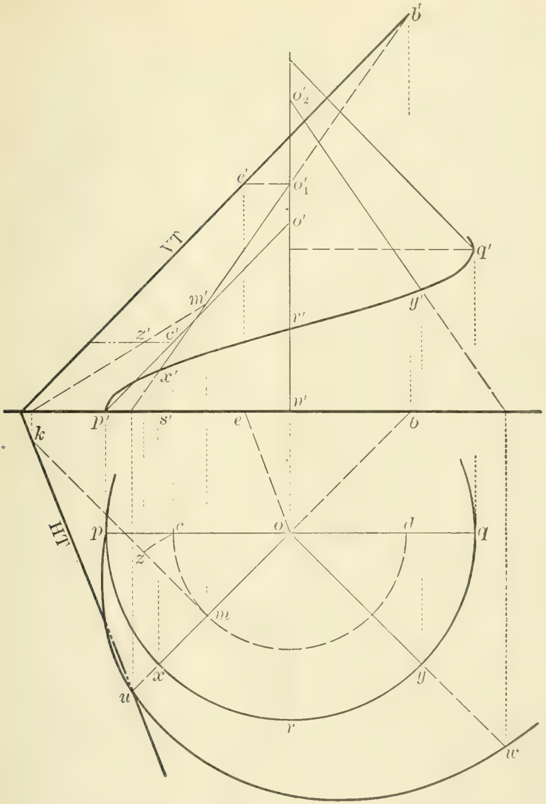

FIG. 71.

152. To assume any point on the surface, we first assume its horizontal projection, as m, and construct the two projections of the element, as (ox, o'_1x'). The vertical projection of the point will then be found on the vertical projection of the element.

If the *vertical* projection of the point is first assumed, and if it does not lie on an element already drawn, it will be necessary to follow the method of Art. 138.

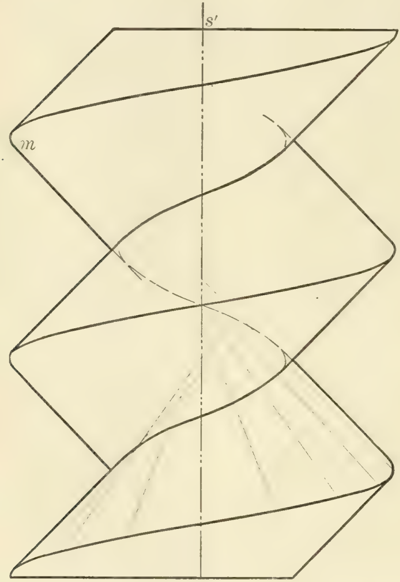

Fig. 72.—Screw-thread. Helicoidal Surface.

153. To construct the horizontal trace of the surface, we find the horizontal piercing points, *p*, *u*, *w*, etc., of the several elements, and the curve *puw* is the horizontal trace of the surface. This curve is the *Spiral of Archimedes.*

The surface is manifestly a warped surface, since, from the nature of its generation, no two consecutive positions of the generatrix can be parallel or intersect.

This surface is an important one, as it forms the curved surface of the thread of the ordinary screw (Fig. 72).

If the generatrix is perpendicular to the axis, the helicoid becomes a particular case of the right conoid (Art. 134), the horizontal plane being the plane directer, and any helix the curvilinear directrix. This is the surface in the squarethreaded screw. The joints in certain types of skew arches are also of this form.

154. Problem 54. **To pass a plane tangent to a helicoid at a point of the surface.**

Let the surface be given as in Fig. 71 and let M be the given point.

Analysis. The tangent plane must contain the rectilinear element passing through the given point, and also the tangent to the helix at this point (Art. 86). The plane of these two lines will then be the required plane.

Construction. MX is the rectilinear element through M. It pierces H at *u*; *cmd* is the horizontal projection of the helix through M. Draw the tangent to this helix at M, as in Art. 73. *mz* will be its horizontal projection, Z the point in which it pierces the horizontal plane through C, and *m'z'* its vertical projection. This tangent pierces H at *k*; hence *ku* is the horizontal trace of the required plane, and VT is the vertical trace.

155. Problem 55. **To pass a plane tangent to a helicoid and perpendicular to a given straight line.**

Analysis. If, with any point of the axis as a vertex, we construct a cone whose rectilinear elements shall make with the horizontal plane the same angle as that made by the rectilinear elements of the given surface, and through the vertex of this cone pass a plane perpendicular to the given line (Art. 44), it will, if the problem be possible, cut from the cone two elements, each of which will be parallel to a rectilinear element of the helicoid and have the same horizontal projection. If through either of these elements of the helicoid a plane be passed parallel to the auxiliary plane, it will be tangent to the surface (Art. 141) and perpendicular to the given line.

Let the problem be constructed in accordance with the analysis.

156. PROBLEM 56. **To find the intersection of a helicoid by a plane.** Let the surface be given as in Art. 150. Intersect by a system of auxiliary planes through the axis. These will cut from the surface rectilinear elements, and from the plane straight lines, the intersection of which will be points of the required curve.

Let the construction be made, and the curve and its tangent represented in true dimensions.

EX. 116. To find the point in which a given line pierces a helicoid. Analyze and construct.

EX. 117. Assume the vertical projection of a point on a helicoid, and find its horizontal projection (Art. 138).

EX. 118. Pass a plane through a given straight line and tangent to a helicoid, and find the point of tangency (Art. 141). When is the problem impossible?

EX. 119. Find the intersection between a helicoid and a cylinder whose elements are parallel to the axis of the helicoid. Analyze.

EX. 120. Find the intersection between a helicoid and an oblique cone — axis of helicoid perpendicular to H, vertex of cone in axis of helicoid, and base of cone lying in V. Analyze.

WARPED SURFACES WITH THREE LINEAR DIRECTRICES

157. A third class of warped surfaces consists of those which may be generated by moving a straight line so as to touch three lines as directrices, or *warped surfaces with three linear directrices.*

To construct a rectilinear element of this class of surfaces passing through a given point on one of the directrices, draw through this point a system of straight lines intersecting either of the other directrices; these form the surface of a cone which will be pierced by the third directrix in one or more points, through which and the given point straight lines may be drawn that will touch the three directrices, and be required elements.

Construction. Let MN, OP, and QR, Fig. 73, be any three linear directrices, and M a given point on the first.

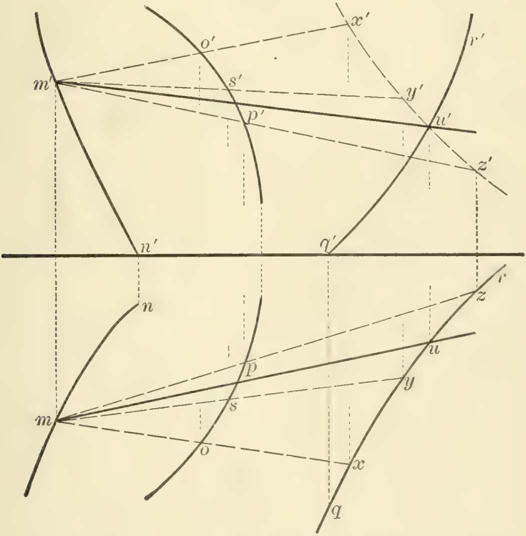

Fig. 73.

Through M draw the lines MO, MS, MP, etc., intersecting OP in O, S, P, etc. They pierce the horizontal projecting cylinder of QR in X, Y, Z, etc., forming a curve, XYZ, which intersects QR in U, the point in which QR pierces the surface of the auxiliary cone. MU is then a required element, two or more of which would be determined if XZ should intersect QR in more than one point.

158. Classes of warped surfaces with three linear directrices. The directrices may all be curved lines, or two of them may be curved and one straight, or one may be curved and two straight, or all three may be straight. The last-named surface is the simplest and most common, and is called a *hyperboloid of one nappe*, as many of its intersections by planes are hyperbolas, while the surface itself is unbroken. For applications of the first two of the above-named surfaces, see the discussion of the gate-recess and the cow's-horn arch in any work on Stereotomy.

159. To construct a rectilinear element of a hyperboloid of one nappe, the method of Fig. 73 is here much simplified, as the auxiliary cone becomes a plane; and the point in which this plane is pierced by the third directrix is found as in Art. 42 or 43.

160. PROPOSITION XLI. **If two warped surfaces, having three directrices, have a common element and three common tangent planes, the points of contact being on this element, they will be tangent all along this element.** For if through each point of contact any intersecting plane be passed, it will intersect the surfaces in two lines, which, besides the given point of contact, will have a second consecutive point in common. If the common element be moved upon the three lines cut from either surface as directrices, to its consecutive position, so as to contain the second consecutive points, it will evidently lie in both surfaces; hence the two surfaces contain two consecutive rectilinear elements and will be tangent all along the common element.

161. To pass a plane tangent to a warped surface with three curvilinear directrices at a point on the surface, a tangent may be drawn to each at the point in which the rectilinear element through the given point intersects it; and then this element may be moved on these three tangents as directrices, generating a hyperboloid of one nappe (Art. 158), which will be tangent to the given surface all along a common element (Art. 160). A plane tangent to this auxiliary surface at the given point will also be tangent to the given surface.

162. An infinite number of planes may, in general, be passed through a point without a warped surface, and tangent to it. For if, through the point, a system of planes be passed intersecting the surface, tangents may be drawn from the point to the curves of intersection, and these will form the surface of a cone tangent to the warped surface. Any plane tangent to this cone will be tangent to the warped surface, and pass through the point.

Also, an infinite number of planes may, in general, be passed tangent to a warped surface and parallel to a straight line. For if the surface be intersected by a system of planes parallel to the line, and tangents parallel to the line be drawn to the sections, they will form the surface of a cylinder tangent to the warped surface. Any plane tangent to this cylinder will be tangent to the warped surface and parallel to the given line.

163. Other varieties of warped surfaces may be generated by moving a straight line so as to touch two lines, having its different positions, in succession, parallel to the different rectilinear elements of a cone; or by moving it parallel to a given plane, so as to touch two surfaces, or one surface and a line; or so as to touch three surfaces, two surfaces and a line, one surface and two lines; or in general, so as to fulfill *any three reasonable conditions.*

It should be remarked that in all these cases the directrices should be so chosen that the surface generated will be neither a plane nor a single-curved surface. Also, that these surfaces, as all others, may be generated by curves moved in accordance with a law peculiar to each variety.

164. If a curve be moved in any way so as not to generate either a plane, a single-curved surface, or a warped surface, it will generate a *double-curved surface*, the simplest variety of which is the *surface of a sphere.*

For a general classification of surfaces, the student is referred to the arrangement on page 76.

Surfaces of Revolution

165. **A surface of revolution** is a surface which may be generated by revolving a line about a straight line as an axis (Art. 34).

From the nature of this generation it is evident that any intersection of such a surface by a plane perpendicular to the axis is the circumference of a circle.

Meridian line and meridian plane. If the surface be intersected by a plane passing through the axis, the line of intersection is a *meridian line*, and the plane a *meridian plane;* and it is also evident that all meridian lines of the same surface are equal, and that the surface may be generated by revolving any one of these meridian lines about the axis.

166. Proposition XLII. **If two surfaces of revolution having a common axis intersect, the line of intersection must be the circumference of a circle whose plane is perpendicular to the axis and whose center is in the axis.** For if a plane be passed through any point of the intersection and the common axis, it will cut from each surface a meridian line (Art. 165), and these meridian lines will have the point in common. If these lines be revolved about the common axis, each will generate the surface to which it belongs, while the common point will generate the circumference of a circle common to the two surfaces, and therefore their intersection. Should the meridian lines intersect in more than one point, the surfaces will intersect in two or more circumferences.

167. **The cylinder of revolution.** The simplest curved surface of revolution is that which may be generated by a straight line revolving about another straight line to which it is parallel. This is evidently a cylindrical surface (Art. 79), and if the plane of the base be perpendicular to the axis, it is a *right cylinder with a circular base.*

The cone of revolution. If a straight line be revolved about another straight line which it intersects, it will generate a conical surface (Art. 83), which is evidently a *right cone*, the axis being the line with which the rectilinear elements make equal angles.

These are the only two single-curved surfaces of revolution.

THE HYPERBOLOID OF REVOLUTION OF ONE NAPPE

168. If a straight line be revolved about another straight line not in the same plane with it, it will generate a surface of revolution called a *hyperboloid of revolution of one nappe.*

To prove that this is a warped surface, let us take the horizontal plane perpendicular to the axis, and the vertical plane parallel to the generatrix in its first position, and let c, Fig. 74, be the horizontal, and $c's'$ the vertical projection of the axis, and MP the generatrix; cm will be the horizontal, and $c'm'$ the vertical projection of the shortest distance between these two lines (Art. 53).

As MP revolves about the axis, CM will remain perpendicular to it, and M will describe a circumference which is horizontally projected in mxy, and vertically in $y'x'$; and as CM is horizontal, its horizontal projection will be perpendicular to the horizontal projection of MP in all of its positions (Prop. XX, Art. 14), and remain of the same length; hence the horizontal projection of MP in any position will be tangent to the circle mxy. No two consecutive positions can therefore be parallel. Neither can they intersect; for from the nature of the motion any two must be separated at any point by the elementary arc of the circle described by that point. *The surface is therefore a warped surface.*

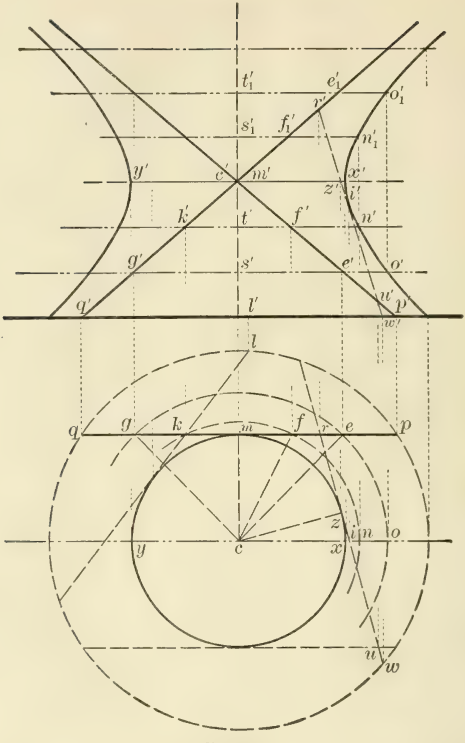

FIG. 74.

The point P, in which MP pierces H, generates the circle *pwq*, which may be regarded as the *base* of the surface.

The circle generated by M is the smallest circle of the surface and is the *circle of the gorge*.

169. To assume a rectilinear element, we take any point in the base, *pwq*, and through it draw a tangent to *xmy*, as *wz* ;

this will be the horizontal projection of an element. Through z erect the perpendicular zz'; z' will be the vertical projection of the point in which the element crosses the circle of the gorge, and $w'z'$ will be the vertical projection of the element.

To assume a point of the surface, we may first assume its horizontal projection, as u, and through it draw the horizontal projection of an element. The vertical projection of the point will be found on the vertical projection of the element.

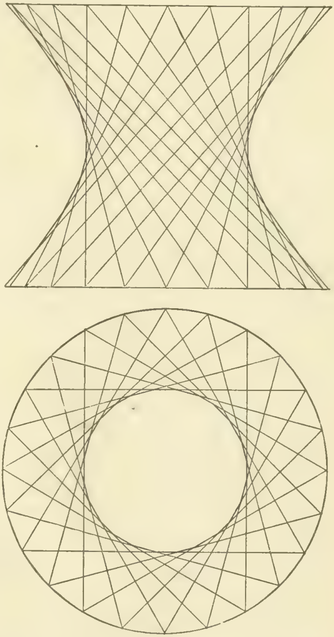

FIG. 75.—Hyperboloid of Revolution of one Nappe, showing Double Generation.

170. Double generation. If through the point M a second straight line as MQ be drawn parallel to the vertical plane, and making with the horizontal plane the same angle as MP, and this line be revolved about the same axis, it will generate the same surface. For if any plane be passed perpendicular to the axis, as the plane whose vertical trace is $e'g'$, it will cut MP and MQ in two points E and G, equally distant from the axis, and these points will, in the revolution of MP and MQ, generate the same circumference; hence the two surfaces must be identical. The surface having two generations by different straight lines, it follows that *through any point of the surface two rectilinear elements can be drawn.*

171. Proof that this surface is a hyperboloid of one nappe. Since the points E and G generate the same circumference, it follows that as MP revolves, MQ remaining fixed, the point E will, at some time of its motion, coincide with G, and the generatrix, MP, intersect MQ in G. In the same way any other point, as F, will come into the point K of MQ, giving another element of the first generation, intersecting MQ at K; and so for each of the points of MP in succession. In this case *kl* will be the horizontal projection of the element of the first generation. Hence, *if the generatrix of either generation remains fixed, it will intersect all the elements of the other generation.*

If, then, any three elements of either generation be taken as directrices, and an element of the other generation be moved so as to touch them, it will generate the surface. It is therefore a *hyperboloid of one nappe* (Art. 158).

A hyperboloid of revolution of one nappe may thus be generated by *moving a straight line so as to touch three other straight lines*, equally distant from a fourth (the axis), the distances being measured in the same plane perpendicular to the fourth, and the directrices making equal angles with this plane.

172. To represent the surface when the first position of the generatrix is not parallel to V, the circle of the gorge may be constructed thus: Let WZ be the first position of the generatrix, and through *c* draw *cz* perpendicular to *wz*: it will be the horizontal projection of the radius of the required circle. The point, of which *z* is the horizontal projection, is vertically projected at z', and *mzx* will be the horizontal, and $y'z'x'$ the vertical projection of the circle of the gorge. With *c* as a center and *cw* as a radius (*w* being the horizontal piercing point of the line), describe the base *pwq*, and the surface will be fully represented.

173. To construct a meridian curve of this surface, we pass a plane through the axis parallel to the vertical plane. It will intersect the horizontal circles generated by the different points of the generatrix in points of the required curve, which will be vertically projected into a curve equal to itself (Art. 58).

Thus the horizontal plane whose vertical trace is $e'y'$ intersects the generatrix in E, and eo is the horizontal projection of the circle generated by this point, and O is the point in which this circle pierces the meridian plane. In the same way the points whose vertical projections are n', x', n'_1, o'_1, etc., are determined.

174. Proof that the meridian curve is a hyperbola. The plane whose vertical trace is $e'_1 o'_1$ at the same distance from $y'x'$ as $e'o'$, evidently determines a point o'_1, at the same distance from $y'x'$ as o'; hence the chord $o'o'_1$ is bisected by $y'x'$, and the curve $o'x'o'_1$ is symmetrical with the line $y'x'$.

The distance $e'o'$, equal to $ce - me$, is the difference between the hypotenuse ce and base me of a right-angled triangle having the altitude mc. As the point o' is further removed from x', the altitude of the corresponding triangle remains the same, while the hypotenuse and base both increase. If we denote the altitude by a, and the base and hypotenuse by b and h respectively, we have

$$h^2 - b^2 = a^2, \text{ and } h - b = \frac{a^2}{h + b},$$

from which it is evident that the difference $e'o'$ continually diminishes as the point o' recedes from x'; that is, the curve $x'n'o'$ continually approaches the lines $p'm'$ and $q'm'$, and will touch them at an infinite distance. These lines are then asymptotes to the curve (Art. 69).

If, now, any element of the first generation, as the one passing through I, be drawn, it will intersect the element of the second generation MQ in R, and the corresponding element on the opposite side of the circle of the gorge in U.

Since this element has but one point in common with the meridian curve, and no point of it or of the surface can be vertically projected on the right of this curve, the vertical projection $u'r'$ must be tangent to the curve at i'. But since ri is equal to iu, $r'i'$ must be equal to $i'u'$, or the part of the tangent included between the asymptotes *is bisected at the point of contact.* This is a property peculiar to the hyperbola, and the meridian curve is therefore a hyperbola; YX being its transverse axis, and the axis of the surface its conjugate (Art. 69). If this hyperbola be revolved about its conjugate axis, it will generate the surface (Art. 165); hence its name.

This is the only warped surface of revolution.

175. Proposition XLIII. **A plane tangent to a surface of revolution is perpendicular to the meridian plane passing through the point of contact.** For this tangent plane contains the tangent to the circle of the surface at this point (Art. 86), and this tangent is perpendicular to the radius of this circle, and also to a line drawn through the point parallel to the axis, since it lies in a plane perpendicular to the axis; and therefore it is perpendicular to the plane of these two lines, which is the meridian plane.

176. Problem 57. **To pass a plane tangent to a hyperboloid of revolution of one nappe at a given point of the surface.**

Let MP, Fig. 76, be one position of the generatrix, and o the horizontal projection of the given point on the surface.

Analysis. Since through the given point two rectilinear elements can be drawn (Art. 170), we have simply to construct these two elements, and pass a plane through them (Art. 142).

Construction. The point M is the nearest to the axis. It therefore generates the gorge circle, horizontally projected in mzs, and vertically projected in a straight line through m' parallel to the ground line (Art. 172). The horizontal pierc-

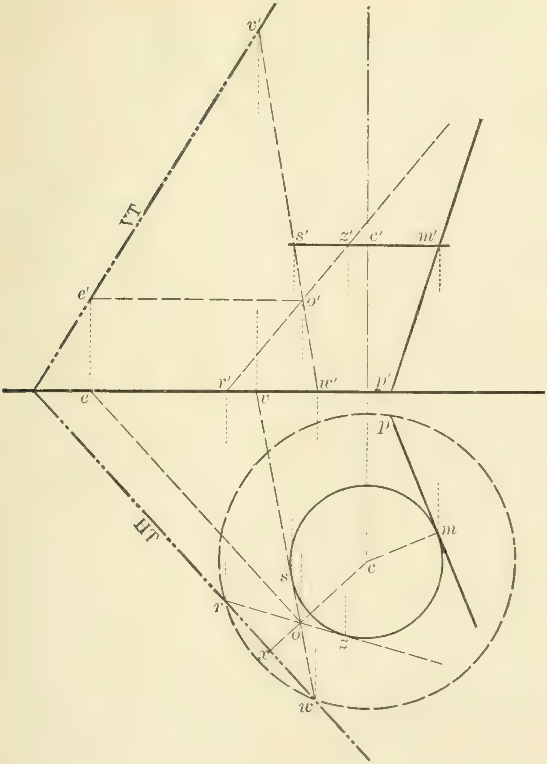

Fig. 76.

ing point *p* generates the circle of the base. Now from the nature of the surface it is evident that *o* is the horizontal projection of two points of the surface — one below, and one above the gorge. In this problem we will consider O to be the point *below* the gorge. If we revolve MP about the axis until it contains the point O, it will take the position *ws*, *w's'*. This is then the element of the first generation through O (Art. 169), and *zr*, *z'r'* will be the element of the second generation (Art. 170). These elements pierce H at *w* and *r*, and *wr* is the horizontal trace of the required plane, and *e'v'* its vertical trace.

Under the assumption that the point O is *above* the gorge, where will the elements of the first and second generations pierce H? Let the student construct the vertical projections of the elements and the traces of the tangent plane.

Since the meridian plane through O must be perpendicular to the tangent plane (Art. 175), its trace *cx* must be perpendicular to *wr*.

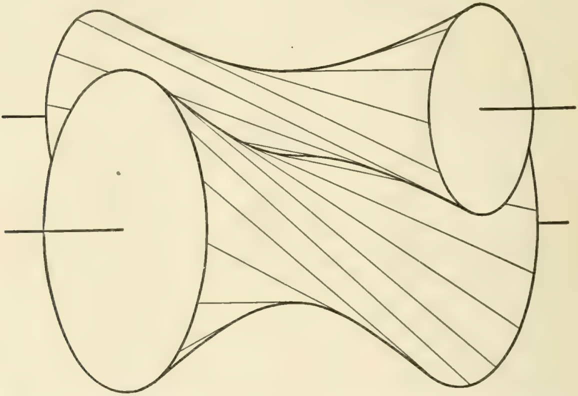

FIG. 77. — Hyperboloids of Revolution Tangent along a Rectilinear Element.

The hyperboloid of revolution of one nappe is, for the sake of brevity, sometimes called the *hypoid*. It is of importance in connection with the design and construction of skew bevel gears. For a further discussion of the properties of this surface, see Rankine's Machinery and Millwork, Arts. 84 and 106, and McCord's Kinematics of Machinery, Arts. 151–170, 378 and 379.

Ex. 121. Given the axis of a hyperboloid of revolution of one nappe, perpendicular to H, and any position of the generating line,

(*a*) Construct the horizontal trace of the surface.

(*b*) Construct the projections of the gorge circle.

(*c*) Construct the element MN through any point, X, of the surface whose horizontal projection is given, it being assumed that the point is *above* the gorge circle.

(*d*) Construct the element OP through any point, Z, of the surface, whose vertical projection is given, it being assumed that the point is on that part of the surface nearer V.

(*e*) Construct the element RS, of the second generation, through X or Z.

(*f*) Construct the vertical projection of the meridian curve parallel to V.

(*g*) Pass a plane, T, tangent to the surface at the point X or Z.

(*h*) Assume a straight line, EF, and find the point, D, in which the surface is pierced by the line.

(*i*) Pass a plane S through the line EF and tangent to the surface.

(*j*) Pass a plane U tangent to the surface and parallel to an assumed plane W, and find the point of tangency, C. (For a hint, see Art. 155.)

(*k*) Construct the curve of intersection with an oblique plane, a cylinder, cone, convolute, any warped surface with plane directer, any warped surface with three linear directrices, or a helicoid.

(*l*) Construct a line tangent to the curve of intersection at an assumed point on the curve.

DOUBLE-CURVED SURFACES OF REVOLUTION

177. The most simple double-curved surfaces of revolution are:

I. A *spherical surface*, or *sphere*, which may be generated by revolving the circumference of a circle about its diameter.

II. An *ellipsoid of revolution*, or *spheroid*, which may be generated by revolving an ellipse about either axis. When the ellipse is revolved about the transverse axis, the surface is a *prolate spheroid;* when about the conjugate axis, an *oblate spheroid.*

III. A *paraboloid of revolution*, which may be generated by revolving a parabola about its axis.

IV. A *hyperboloid of revolution of two nappes*, which may be generated by revolving a hyperbola about its transverse axis.

V. A *torus*, which may be generated by revolving a circumference about a line in the plane of and outside the circumference.

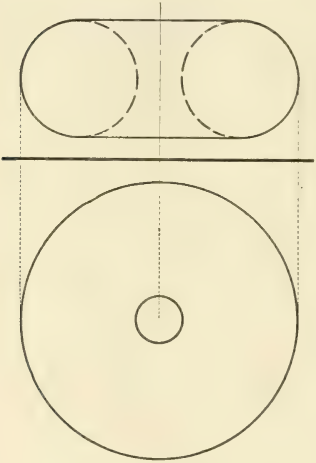

FIG. 78.—Torus.

178. These surfaces of revolution are usually represented by taking the horizontal plane perpendicular to the axis, and then drawing the intersection of the surface with the horizontal plane; or by taking the horizontal projection of the greatest horizontal circle of the surface for the horizontal projection, and then projecting on the vertical plane the meridian line which is parallel to that plane for the vertical projection.

179. To assume a point on a surface of revolution, we first assume either projection as the horizontal, and erect a perpendicular to the horizontal plane. Through this perpendicular pass a meridian plane; it will cut from the surface a meridian line which will intersect the perpendicular in the required point or points.

Construction. Let the surface be a prolate spheroid, Fig. 79, and let c be the horizontal, and $c'd'$ the vertical projection of the axis, mon the horizontal projection of its largest circle, and $d'm'c'n'$ the vertical projection of the meridian curve parallel to the vertical plane, and let p be the assumed horizontal projection; p will be the horizontal, and $s'p'$ the vertical projection of the perpendicular to H; cp will be the horizontal trace of the auxiliary meridian plane. If this plane be now revolved about the axis until it becomes parallel to the vertical plane, p will describe the arc pp_1, and p_1 will be the horizontal, and $p'_1q'_1$ the vertical projection of the revolved position of the perpendicular. The meridian curve, in its revolved position, will be vertically projected into its equal $d'm'c'n'$, and p'_1 and q'_1 will be the vertical projections of the two points of intersection in their revolved position. When the meridian plane is revolved to its primitive position, these points will describe the arcs of horizontal circles, projected on H in p_1p, and on V in p'_1p' and q'_1q', and p' and q' will be the vertical projections of the two points of the surface horizontally projected in p.

Let the student assume the vertical projection of a point on the surface, and find its horizontal projection.

Also let the student represent a point on the surface of the torus, Fig. 78, taking the horizontal projection about an eighth of an inch from the inner circle.

PROBLEMS RELATING TO PLANES TANGENT TO SURFACES OF REVOLUTION

180. These problems are, in general, solved either by a direct application of the rule in Art. 86, taking care to intersect the surface by planes, so as to obtain the two simplest curves of the surface intersecting at the point of contact, or by means of more simple auxiliary surfaces tangent to the given surface.

181. PROBLEM 58. **To pass a plane tangent to a sphere at a given point.**

Analysis. Since the radius of the sphere, drawn to the point of contact, is perpendicular to the tangent to any great circle at this point, and since these tangents all lie in the tangent plane (Art. 86), *this radius must be perpendicular to the tangent plane.* We have then simply to pass a plane perpendicular to this radius at the given point, and it will be the required plane.

Construction. Let the point be assumed as in Art. 179, and the plane passed as in Art. 44. The construction is left for the student.

182. PROBLEM 59. **To pass a plane tangent to any surface of revolution at a given point.**

Let the surface be given as in Art. 179, Fig. 79, and let P be the point.

Analysis. If at the given point we draw a tangent to the meridian curve of the surface, and a second tangent to the circle of the surface at this point, the plane of these two lines will be the required plane (Art. 86).

Construction. Through P pass a meridian plane; cp will be its horizontal trace. Revolve this plane about the axis of the surface until it is parallel to V. P_1 will be the revolved position of the point of contact. At p'_1 draw p'_1y' tangent to $c'n'd'm'$ (Art. 67). It will be the vertical projection of the revolved position of a tangent to the meridian curve at P.

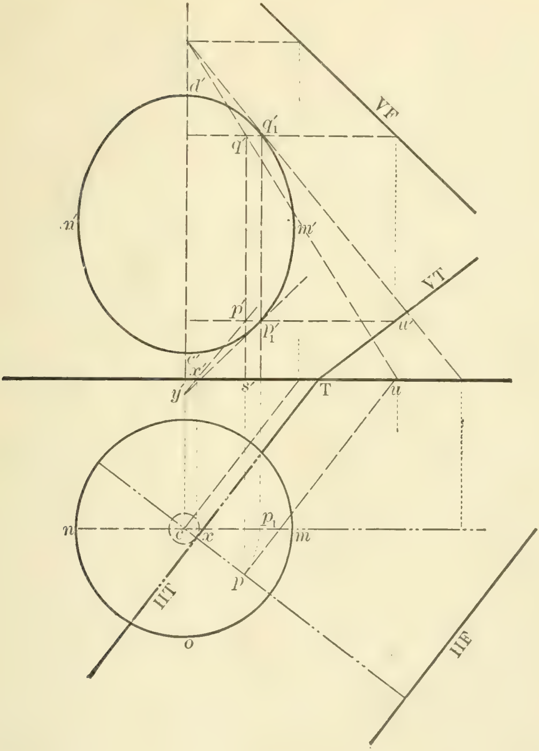

Fig. 79.

When the plane is revolved to its primitive position, the point, of which y' is the vertical projection, remains fixed, and $y'p'$ will be the vertical projection of the tangent, and cp its horizontal projection. It pierces H at x, one point of the horizontal trace of the required plane. pp_1 is the horizontal, and $p'p'_1$

the vertical projection of an arc of the circle of the surface containing P. At p draw pu perpendicular to cp. It will be the horizontal projection of the tangent to the circle at P, and $p'u'$ is its vertical projection. This pierces V at u', a point of the vertical trace. Through x draw xT parallel to pu, and through u' draw u'T; these will be the traces of the required plane.

By the same method we may pass a plane tangent to any other surface of revolution at a given point.

Since the tangent plane and horizontal plane are both perpendicular to the meridian plane through the point of contact, their intersection, which is the horizontal trace, will be perpendicular to the meridian plane and to its horizontal trace.

While at a given point on a double-curved surface only one tangent plane can be passed, it may be proved, as in Art. 162, that from a point without the surface an infinite number of such planes can be passed.

183. PROBLEM 60. **To pass a plane through a given straight line and tangent to a sphere.**

Analysis. Conceive the sphere to be circumscribed by a cylinder of revolution whose axis is parallel to the given line. The line of contact will be the circumference of a great circle whose plane is perpendicular to the axis and to the given line. A plane through the straight line tangent to this cylinder will be tangent also to the sphere. The plane of the circle of contact will intersect the given line in a point, and the required tangent plane in a straight line drawn from this point tangent to the circle. The plane of this tangent and the given line will be the required plane. Without constructing the cylinder, we have then simply to *pass a plane through the center of the sphere perpendicular to the given line, and from the point in which it intersects the line, to draw a tangent to the circle cut from the sphere by the same plane, and pass a plane through this tangent and the given line.*

Construction. The student will assume a sphere and an oblique line outside the sphere, and make the construction in accordance with the above analysis.

184. PROBLEM 61. **To pass a plane through a given straight line and tangent to any surface of revolution.**

Let the horizontal plane be taken perpendicular to the axis, of which c, Fig. 80, is the horizontal, and $c'd'$ the vertical, projection. Let poq be the intersection of the surface by the horizontal plane, $p'd'q'$ the vertical projection of the meridian curve parallel to the vertical plane, and MN the given line.

Analysis. If this line revolve about the axis of the surface, it will generate a hyperboloid of revolution of one nappe (Art. 168), having the same axis as the given surface. If we now conceive the plane to be passed through the line tangent to the surface, it will also be tangent to the hyperboloid at a point of the given straight line (Art. 141); and since the meridian plane through the point of contact on each surface must be perpendicular to the common tangent plane (Art. 175), these meridian planes must form one and the same plane. This plane will cut from the given surface a meridian curve, from the hyperboloid a hyperbola (Art. 174), and from the tangent plane a straight line tangent to these curves at the required points of contact (Art. 86). The plane of this tangent and the given line will therefore be the required plane.

Construction. Construct the hyperboloid, as in Arts. 172 and 173. yx will be the horizontal, and $x'y'w'$ the vertical, projection of one branch of the meridian curve parallel to V. If the meridian plane through the required points of contact be revolved about the common axis until it becomes parallel to V, the corresponding meridian curves will be projected, one into the curve $p'd'q'$, and the other into the hyperbola $x'y'w'$. Tangent to these curves draw $x'r'$; X will be the revolved position of the point of contact on the hyperbola (found as in Art. 69), and R that on the meridian curve of the given surface (Art. 67). When the meridian plane is revolved to its true position, X will be horizontally projected in *mn* at *s*; *sc* will be the horizontal trace

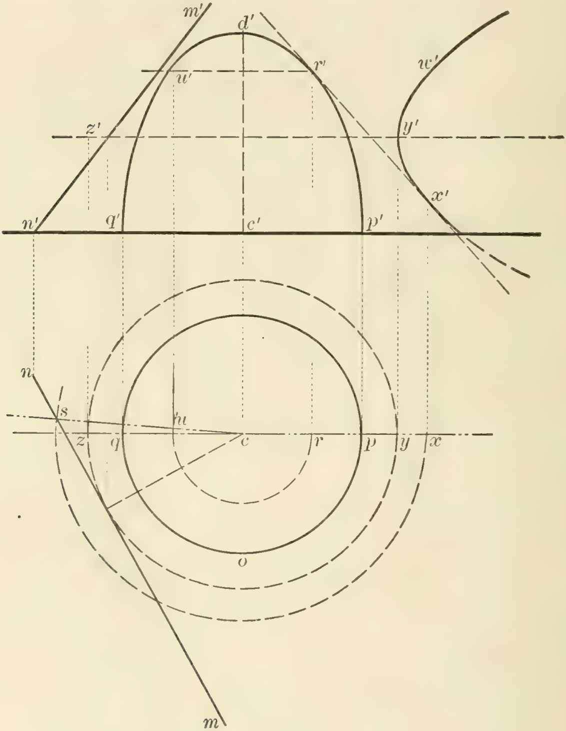

Fig. 80.

of the meridian plane, and *u* will be the horizontal, and *u'* the vertical, projection of the point of contact on the given surface. A plane through this point and MN will be the tangent plane.

185. In general, through a given straight line a limited number of planes only can be passed tangent to a double-curved surface. For let the surface be intersected by a system of planes parallel to the given line, and tangents be drawn to the sections also parallel to the line. These will form a cylinder tangent to the surface. Any plane through the straight line tangent to this cylinder will be tangent to the surface; and the number of tangent planes will be determined by the number of tangents which can be drawn from a point of the given line to a section made by a plane through this point.

INTERSECTION OF SURFACES OF REVOLUTION WITH OTHER SURFACES

186. PROBLEM 62. **To find the intersection of any surface of revolution by a plane.**

Let the surface be a hyperboloid of revolution of one nappe given as in Fig. 81, and let T be the cutting plane.

Analysis. If a meridian plane be passed perpendicular to the cutting plane, it will intersect it in a straight line, which will divide the curve symmetrically, and be an axis. If the points in which this line pierces the surface be found, these will be the points on the axis of symmetry of the curve. Now intersect by a system of planes perpendicular to the axis of the surface; each plane will cut from the surface a circumference, and from the given plane a straight line, the intersection of which will be points of the required curve.

Construction. **To find the points on the axis of symmetry,** draw *cn* perpendicular to HT; it will be the horizontal trace of the auxiliary meridian plane. This plane intersects T in the straight line NC (Art. 40), and the surface in a meridian curve which is intersected by NC in the two vertices of the required curve.

To find these points, revolve the meridian plane about the axis until it becomes parallel to V. The meridian curve will be vertically projected into the hyperbola which limits the vertical projection of the surface (Art. 58), and the line NC into *d′c′*. The points *s′* and *r′* will be the vertical projections

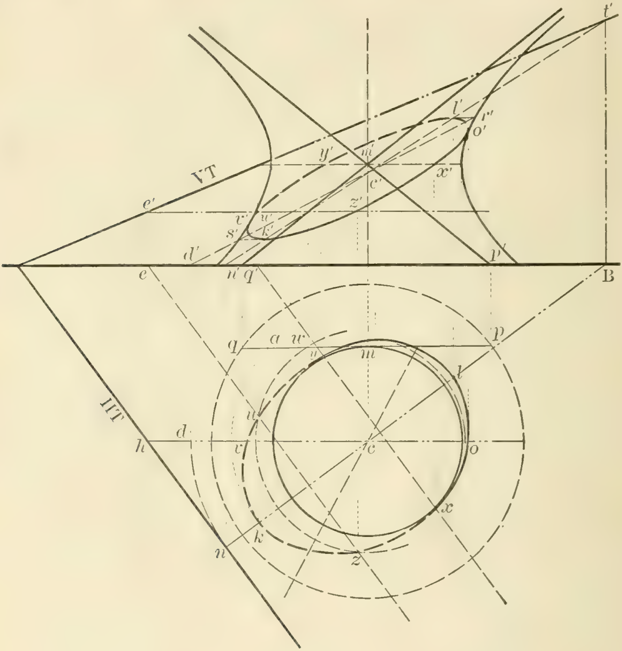

FIG. 81.

of the revolved positions of the vertices. After the counter-revolution, these points are horizontally projected at *k* and *l*, and vertically at *k′* and *l′*.

To find the points in which the vertical projection of the curve of intersection is tangent to the contour line of the surface, use the meridian plane parallel to V. It cuts from the given plane a line horizontally projected in *ho* and vertically projected in a line (not shown) parallel to VT. It cuts from the surface of revolution the meridian curve whose vertical projection is already drawn. These lines intersect in the points *vv′* and *oo′*, the required points of tangency.

To find other points of the curve, let *e′z′* be the vertical trace of an auxiliary plane perpendicular to the axis. It cuts the surface in a circle horizontally projected in *wuz*, and the plane in a straight line, which piercing V at *e′* is horizontally projected in *ez*; hence U and Z are points of the curve. In the same way any number of points may be determined.

The points upon any particular circle may be determined by using the plane of this circle as an auxiliary plane. If the curve crosses the circle of the gorge, the points in which it crosses are determined by using the plane of this circle, and in this case the horizontal projection of the curve must be tangent to the horizontal projection of the circle of the gorge, at the points *x* and *y*.

To draw a tangent to the curve at Z, pass a plane tangent to the surface at Z, as in Art. 182; its intersection with T will be the tangent (Art. 105). The curve may be represented in its true dimensions as in Art. 110.

Let the intersection of an oblique plane with a sphere, an ellipsoid of revolution, a torus, and a paraboloid of revolution, be constructed in accordance with the principles given above.

187. PROBLEM 63. **To find the intersection of a cylinder and a sphere.**

Analysis. Intersect the surfaces by a system of auxiliary planes parallel to the rectilinear elements of the cylinder and perpendicular to the horizontal plane (or the plane of the base). Each plane will cut from the cylinder two rectilinear elements, and from the sphere a circle, the intersection of which will be points of the required curve.

Construction. Revolve each auxiliary plane about its horizontal trace until it coincides with H. Find the revolved position of each circle and of the corresponding pairs of elements. They will intersect in points which, when returned to their primitive positions, will be points on the required curve of intersection.

A tangent at any point of the curve may be constructed by finding the intersection of two planes, one tangent to the cylinder and one to the sphere at that point, as in Art. 116.

Let the student assume an oblique cylinder and a sphere intersecting it, and solve the problem in accordance with the above method.

188. PROBLEM 64. **To find the intersection of two surfaces of revolution whose axes are in the same plane.**

First, let the axes intersect and let one of the surfaces be an ellipsoid of revolution and the other a paraboloid; and let the horizontal plane be taken perpendicular to the axis of the ellipsoid, and the vertical plane parallel to the axes; (c, $c'd'$), Fig. 82, being the axis of the ellipsoid, and (cl, $s'l'$) that of the paraboloid. Let the ellipsoid be represented as in Art. 179 and let $z'f'r'$ be the vertical projection of the paraboloid.

Analysis. Intersect the two surfaces by a system of auxiliary spheres having their centers at the point of intersection of the axes. Each sphere will intersect each surface in the circumference of a circle perpendicular to its axis (Art. 166), and the points of intersection of these circumferences will be points of the required curve.

Construction. With s' as a center and any radius $s'q'$, describe the circle $q'p'r'$; it will be the vertical projection of an auxiliary sphere. This sphere intersects the ellipsoid in a circumference vertically projected in $p'q'$, and horizontally in pxq. It intersects the paraboloid in a circumference vertically projected in $r'v'$. These circumferences intersect in two points vertically projected at x', y', and horizontally at x and y.

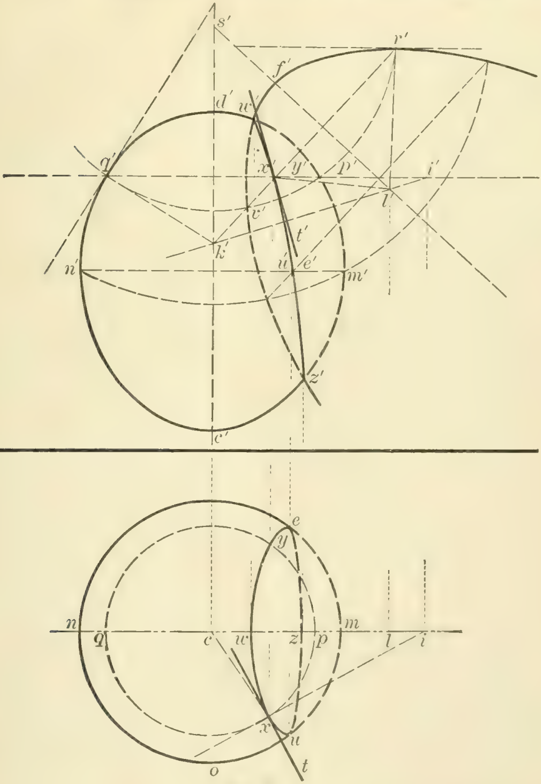

FIG. 82.

In the same way any number of points may be found.

The points on the greatest circle of the ellipsoid are found by using $s'n'$ as a radius. These points are horizontally projected at u and e, points of tangency of xzy with nom.

The points W and Z are the points in which the meridian curves parallel to V intersect, and are points of the required curve.

Each point of the curve $z'x'w'$ is the vertical projection of two points of the curve of intersection, one in front and the other behind the plane of the axes.

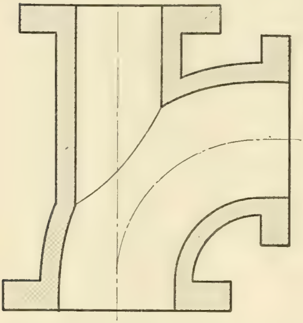

FIG. 83. — Intersection of Cylinder and Torus. Pipe Connection.

A tangent may be drawn to the curve at any point, as X, as in Art. 111.

Second. **If the axes of the two surfaces are parallel**, the construction is more simple, as the auxiliary spheres become planes perpendicular to the axes. Let the construction be made in this case.

Let it be required to find the intersection between the cylinder and the torus of Fig. 83. How would the auxiliary planes be passed?

Ex. 124. Given a torus (Art. 177), and the vertical projection of a point A on the surface.

(*a*) Find the horizontal projection of A, assuming that it lies on the side of the surface toward V.

(*b*) Pass a plane S tangent to the surface at the point A.

(*c*) How would you solve (*b*) if the surface were a sphere?

(*d*) Pass a plane T tangent to the torus and through an assumed straight line, BC.

(*e*) How would you solve (*d*) if the surface were a sphere?

(*f*) Pass a plane, U, tangent to the torus and perpendicular to the line BC.

(*g*) How would you solve (*f*) if the surface were a sphere?

(*h*) Assume the projections of a curved line cutting the surface of the torus, and find the piercing points, P and Q.

(*i*) Find the curve of intersection of the torus with a plane, an oblique cylinder or cone with circular base, a helical convolute, a warped surface with plane directer parallel to H, a helicoid, an ellipsoid of revolution whose axis intersects the axis of the torus, a paraboloid of revolution whose axis is parallel to the axis of the torus.

PROBLEMS RELATING TO TRIHEDRAL ANGLES — GRAPHICAL SOLUTION OF SPHERICAL TRIANGLES

189. In the preceding articles we have all the elementary principles and rules relating to the orthographic projection. The student who has thoroughly mastered them will have no difficulty in their application.

Let this application now be made to the solution of the following simple problems.

190. PROBLEM 65. **Having given two of the faces of a trihedral angle and the dihedral angle opposite one of them, to construct the trihedral angle.**

Let dsf and fse_1, Fig. 84, be the two given faces, and A the given angle opposite fse_1, dsf being in the horizontal plane and the vertical plane perpendicular to the edge sf.

Construct, as in Art. 50, de', the vertical trace of a plane whose horizontal trace is sd and which makes with H the angle A; sde' will be the true position of the required face. Revolve fse_1 about sf until the point e_1 comes into de' at e'. This must be the point in which the third edge in true position pierces V. Join $e'f$; it will be the vertical trace of the plane of the face fse_1 in true position, and SE will be the third edge.

Revolve *sde′* about *sd* until it coincides with H; *e′* falls at e_2 (Art. 34), and dse_2 is the true size of the third or required face.

e′fd is the dihedral angle opposite the face *dse′*, and *pvq*, determined as in Art. 47, is the dihedral angle opposite *dsf*.

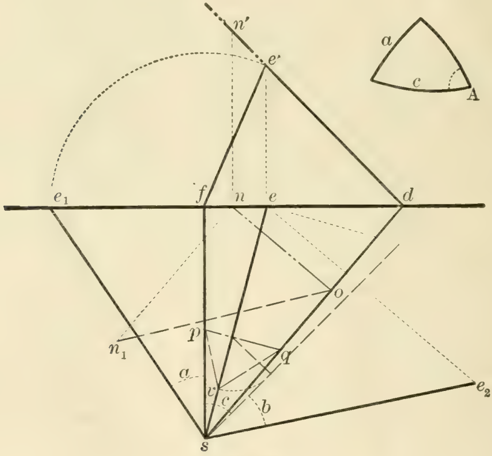

Fig. 84.

191. Problem 66. **Having given two dihedral angles formed by the faces of a trihedral angle, and the face opposite one of them, to construct the angle.**

Let A and B, Fig. 85, be the two dihedral angles, and dse_2 the face opposite B.

Make *e′df* equal to A. Revolve dse_2 about *ds* until e_2 comes into *de′* at *e′*. This will be the point where the edge se_2 in true position pierces V, and SE will be this edge.

Draw $e'm$, making $e'me$ equal to B, and revolve $e'm$ about $e'e$; it will generate a right cone whose rectilinear elements all make with H an angle equal to B. Through s pass the plane sfe' tangent to this cone (Art. 100). It, with the faces fsd and sde', will form the required angle.

fse_1 is the true size of the face opposite A, e_1 being the revolved position of e', determined as in Art. 34, and the third dihedral angle formed by $e'fs$ and $e'ds$ may be found as in Art. 47.

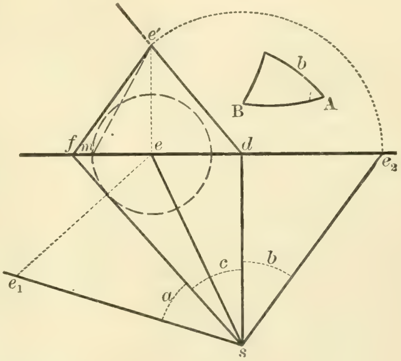

Fig. 85.

192. Problem 67. **Given two faces of a trihedral angle and their included dihedral angle, to construct the angle.**

Let dse_2 and dsf, Fig. 86, be the two given faces and A the given angle.

Make $e'df$ equal to A. de' will be the vertical trace of the plane of the face dse_2 in its true position. Revolve dse_2 about sd until e_2 comes into de' at e', the point where the edge se_2 in true position pierces V. Draw $e'f$. It is the vertical trace of the plane of the third or required face, and fse_1 is its true size.

eoe_3 (Art. 49) is the dihedral angle opposite $e'ds$, and the third dihedral angle can be found as in Art. 47.

193. Problem 68. **Given one face and the two adjacent dihedral angles of a trihedral angle, to construct the angle.**

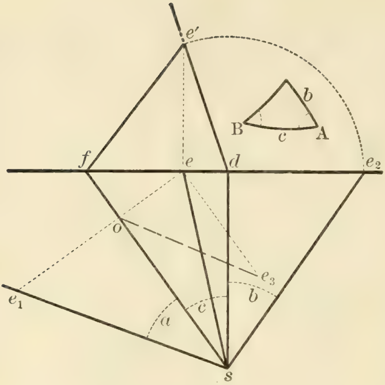

FIG. 86.

Let *dsf*, Fig. 86, be the given face, and A and B the two adjacent dihedral angles.

Make *e′df* equal to A. Construct as in Art. 50, *fe′*, the vertical trace of a plane *sfe′*, making with H an angle equal to B. SE will be the third edge, dse_2 and fse_1 the true size of the other faces, and the third dihedral angle is found as in Art. 47.

194. PROBLEM 69. **Given the three faces of a trihedral angle, to construct the angle.**

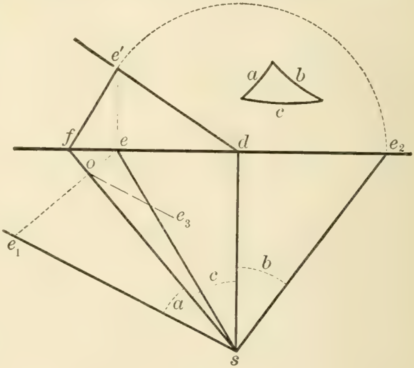

FIG. 87.

Let dse_2, *dsf*, and fse_1, Fig. 87, be the three given faces, *sd* and *sf* being the two edges in the horizontal plane.

Make se_1 equal to se_2. Revolve the face fse_1 about *fs*; e_1 describes an arc whose plane is perpendicular to H (Art. 34), of which e_1e is the horizontal and *ee′* the vertical trace. Also revolve

dse_2 about ds; e_2 describes an arc in the vertical plane. These two arcs intersect at e', the point where the third edge pierces V, and SE is this edge.

Join $e'd$ and $e'f$. These are the vertical traces of the planes of the faces dse_2 and fse_1 in true position. The dihedral angles may now be found as in the preceding articles.

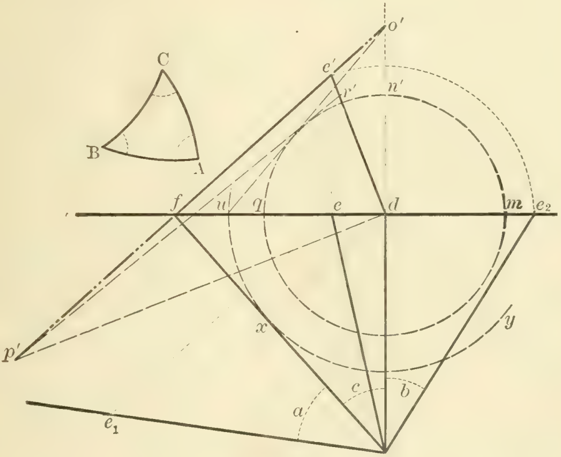

FIG. 88.

195. PROBLEM 70. **Given the three dihedral angles formed by the faces of a trihedral angle, to construct the angle.**

Let A, B, and C, Fig. 88, be the dihedral angles.

Make $e'df$ equal to A. Draw ds perpendicular to the ground line and take $e'ds$ as the plane of one of the faces. If we now construct a plane which shall make with H and $e'ds$ angles respectively equal to B and C, it, with these planes, will form the required trihedral angle.

To do this: with d as a center and any radius as dm, describe a sphere; *mn'q* will be its vertical projection. Tangent to *mn'q* draw *o'u*, making *o'ud* equal to B, and revolve it about *o'd*. It will generate a cone whose vertex is *o'*, tangent to the sphere, and all of whose rectilinear elements make with H an angle equal to B.

Also tangent to *mn'q* draw *p'r'*, making with *de'* an angle equal to C, and revolve it about *p'd*. It will generate a cone whose vertex is *p'*, tangent to the sphere, and all of whose rectilinear elements make with the plane *sde'* an angle equal to C. If now through *o'* and *p'* a plane be passed tangent to the sphere, it will be tangent to both cones and be the plane of the required third face. *p'o'* is the vertical trace of this plane, and *fs* tangent to the base *uxy* is the horizontal trace, SE is the third edge, and *dsf*, dse_2, and fse_1 the three faces in true size.

196. By a reference to Spherical Trigonometry, it will be seen that the preceding six problems are simple constructions of the required parts of a spherical triangle when any three are given. Thus in Problem 65, *two sides a and c, and an angle A opposite one*, are given, and the others constructed. In Problem 66, *two angles A and B and a side b, opposite one*, are given, etc.

197. PROBLEM 71. **To construct a triangular pyramid, having given the base and the three lateral edges.**

Let *cde*, Fig. 89, be the base in the horizontal plane, the ground line being taken perpendicular to *cd*, and let *c*S, *d*S, and *e*S be the three edges. With *c* as a center and *c*S as a radius, describe a sphere, intersecting H in the circle *mon*. The required vertex must be in the surface of this sphere. With *d* as a center and *d*S as a radius, describe a second sphere, intersecting H in the circle *qmp*. The required vertex must also be on this surface. These two spheres intersect in a circle of which *mn* is the horizontal and *m's'n'* the vertical projection (Art. 166). With *e* as a center and *e*S as a radius, describe a third sphere, intersecting the second in a circle of which *qp* is the horizontal projection. These two circles intersect in a straight line perpendicular to H at *s*, and vertically projected in *r′s′*. This line intersects the first circle in S, which must be a point common to the three spheres, and therefore the vertex

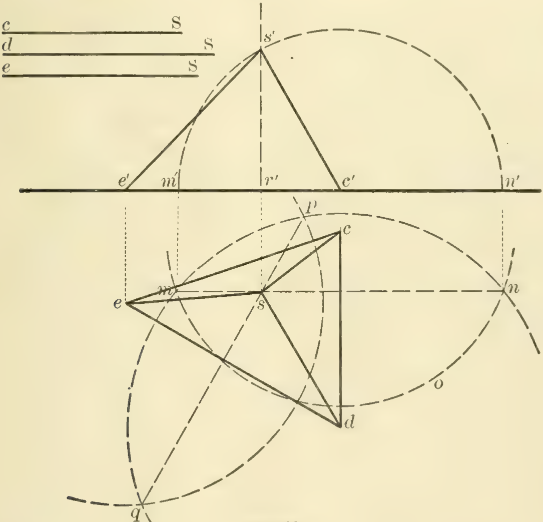

FIG. 89.

of the required pyramid. Join S with C, D, and E, and we have the lateral edges, in true position.

198. PROBLEM 72. **To circumscribe a sphere about a triangular pyramid.**

Let *mno*, Fig. 90, be the base of the pyramid in the horizontal plane, and S its vertex.

Analysis. Since each edge of the pyramid must be a chord of the required sphere, if any edge be bisected by a plane perpendicular to it, this plane will contain the center of the sphere. Hence, if three such planes be constructed intersecting in a point, this must be the required center, and the radius will be the straight line joining the center with the vertex of any trihedral angle.

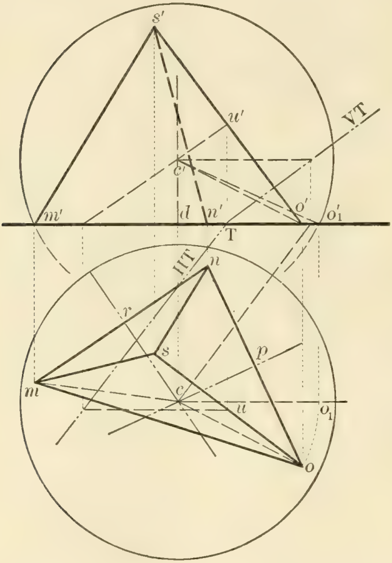

Fig. 90.

Construction. Bisect *mn* and *no*, by the perpendiculars *rc* and *pc*. These will be the horizontal traces of two bisecting and perpendicular planes. They intersect in a straight line perpendicular to H at *c*. Through U, the middle point of SO, pass a plane perpendicular to it (Art. 44). T is this plane. It is pierced by the perpendicular (*c*, *dc'*) at C, which is the intersection of the three bisecting planes, and therefore the center of the required sphere. CO is its radius, the true length of which is $c'o'_1$ (Art. 26).

With *c* and *c'* as centers, and with $c'o'_1$ as a radius, describe circles. They will be, respectively, the horizontal and vertical projections of the sphere.

199. Problem 73. **To inscribe a sphere in a given triangular pyramid.**

Let the pyramid S-*mno*, Fig. 91, be given as in the preceding problem.

Analysis. The center of the required sphere must be equally distant from the four faces of the pyramid, and therefore must be in a plane bisecting the dihedral angle formed by either two of its faces. Hence, if we bisect three of the dihedral angles by planes intersecting in a point, this point must be the center of the required sphere, and the radius will be the distance from the center to any face.

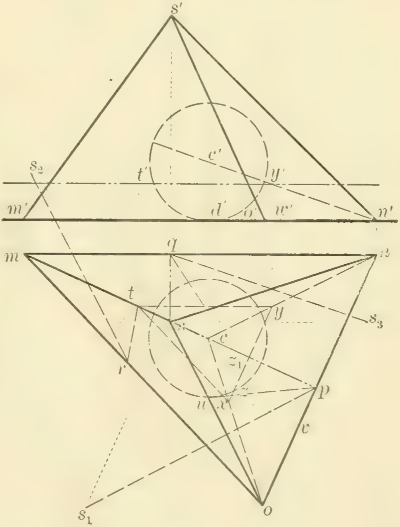

FIG. 91.

Construction. Find the angle sps_1 made by the face S*on* with H (Art. 49). Bisect this by the line *pu*, and revolve the plane sps_1 to its true position. The line *pu*, in its true position, and *on* will determine a plane bisecting the dihedral angle sps_1. In the same way determine the planes bisecting the dihedral angles srs_2, sqs_3. These planes, with the base *mno*, form a second pyramid, the vertex of which is the intersection of the three planes, and therefore the required center.

Intersect this pyramid by a plane parallel to H, whose vertical trace is $t'y'$. This plane intersects the faces in lines parallel to *mn*, *no*, and *om* respectively. These lines form a triangle whose vertices are in the edges. To determine these lines, lay off *pv* equal to $y'w'$; draw *vz* parallel to *ps*. *z* will be the revolved position of the point in which the parallel plane intersects *pu* in its true position. z_1 is the horizontal projection of this point, and z_1y of the line parallel to *no*. In the same way *ty* and *tx* are determined. Draw *ox* and *ny*; they will be the horizontal projections of two of the edges. These intersect in *c*, the horizontal projection of the vertex. $n'y'$ is the vertical projection of the edge which pierces H at *n*, c' the vertical projection of the center, and $c'd'$ the radius.

With *c* and c' as centers, describe circles with $c'd'$ as a radius. They will be the horizontal and vertical projections of the required sphere.

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